Suppose \( f \) is a function such that \( f \) exists at some point \( P \). If you zoom in on the graph, the curve appears more and more like the tangent line to \( f \) at \( P \).
This is why you've had to do so many "find the equation for the tangent line to the given point" problems!

Therefore, it makes sense to approximate a function with its tangent line at \( P \) (at a point \( P \)).

That is, the curve approaches the tangent line that is differentiable at a point \( P \) is one of the properties of a function that is differentiable on smaller scales - it is the basis of linear approximations.

This idea - that smooth curves (i.e., curves without corners) appear straighter on smaller scales - is the basis of linear approximations.
\( \langle x, 0 \rangle \) is the same thing!

\[ (v - x)(v, f) = (v, f - h) \]

So the equation of the tangent line is

\[ (v, f)' \text{, the slope of the line tangent to the curve at point } f \]

At a given point, the following: At a given point.

**Remarks:** Compare this definition to the following: At a given point.

For \( x \) in \( I \)

\[ (v - x)(v, f) + (v, f)' = (x, f)' \]

**Linear approximation to** \( f \) **as the linear function**

**Definition**

**Suppose** \( f \) **is differentiable on an interval** \( I \) **containing the point** \( a \). The

\[ (v, f)' \text{ is the same thing!} \]
\[
\frac{4}{I} + \frac{x}{I} = (1 - x) \frac{4}{I} + \frac{2}{I} = (x)_f
\]

\[
\frac{4}{I} = (v)_f, \quad \frac{2}{I} = (v)_f, \quad \frac{2(1 + x)}{I} = (x)_f
\]

Solution: First compute

Then use the linear approximation to estimate

\[
(1.1, 1.1) \approx (v)_f(1.1) + (v)_f = (x)_f
\]

Approximation to

Write the equation of the line that represents the linear approximation to

Exercise
\[ 0.5238 \times \frac{0.5238}{0.5238 - 0.5238} = 100 = 0.23\% . \]

Note that \( f(1.1) = 0.5238 \), so the error in this estimation is

\[ T(1.1) \approx f(T(1.1)) \]

Because \( x = 1.1 \) is near \( a = 1 \), we can estimate \( f(T(1.1)) \) using

Solution (continued):
Exercise

(a) The linear approximation to \( f(x) = \sqrt{1 + x} \) at the point \( x = 0 \) is (choose one):

- \( A. \quad L(x) = 1 \)
- \( B. \quad L(x) = x \quad (\text{choose one:} \quad f(0) = 1 + \frac{1}{2}(1 + 0.5) \quad \text{or} \quad f(0) = 1 + \frac{1}{2}(1 - 0.5)) \)
- \( C. \quad L(x) = x + \frac{1}{2} \)
- \( D. \quad L(x) = 1 - \frac{x}{2} \)

(b) What is an approximation for \( f(0.1) \)?

\[ f(0.1) = \frac{1}{2} \sqrt{1 + 0.1} \]

\[ \approx 1 + \frac{1}{2}(0.1) \]

\[ 1.05 \]

\[ 1.05 \]
\[ x \nabla (v) f \approx h \nabla \quad \iff \quad (v - x)(v) f \approx (v)f - (x)f \]

When rewritten,

\[ (v - x)(v) f + (v)f \approx (x)f \]

is fixed and \( x \) is a nearby point:

Our linear approximation \( L(x)f \) is used to approximate \((x)f \) when \( a \).
\( I, \text{ is approximately}\)

\[ x \nabla (x), f \]

\( \text{the value of } f \text{ between two points } a \text{ and } a + \Delta x \in J \text{ containing the point } a, \text{ then the} \)

\[ (\alpha, f) \approx \frac{x \nabla}{\Delta x} \]

\[ x \nabla (\alpha), f \approx \Delta x \]

This is another way to say that \((\alpha), f \) is the rate of change of \( y \) with respect to \( x \).

A change in \( y \) can be approximated by the corresponding change in \( x \).

The case for these slices was done by Durbin-Stenberg, later extended to higher dimensions.
\[ dy \approx y \Delta \]

Changes from \( a \) to \( a + \Delta x \) (called the differential, \( dy \)).

The change in the linear approximation \( y = f(x) \) as \( x \) changes from \( a \) to \( a + \Delta x \) (which we call \( y \Delta \)).

The change in the function \( y = f(x) \) as \( x \) changes from \( a \) to \( a + \Delta x \).

We now have two different, but related quantities:
\[ \left( \mathbf{x} \times \nabla + a \right) f = \mathbf{h} \]

and this is:

\[ \nabla = \nabla \mathbf{h} \]

\[ x \nabla (v), f = \]

\[ (v - a)(v), f + (v) f \] - \[ (v - x \nabla + a)(v), f + (v) f \] =

\[ (v) I - (x \nabla + a) I = X \nabla \]

The linear approximation change is:

\[ (v) f - (x \nabla + a) f = \mathbf{h} \nabla \]

The function change is exactly:

\[ \Delta x \nabla + a \] to \[ a \to x \nabla + a \]
We define the differentials $dx$ and $dy$ to distinguish between the change in the function ($\Delta y$) and the change in the linear approximation ($\Delta L$):

- $dy$ is the change in the linear approximation, which is $\Delta L = f'(a) \Delta x$.
- $dx$ is simply the change in $x$, i.e., $\Delta x$.

So:

\[
\frac{dy}{dx} = f'(a) \quad (\text{at } x = a)
\]
The use of differentials is critical as we approach integration.

\[ (x) - (x \Delta x + x) = \Delta x \]

\[ \Delta x = \Delta f = f(x) \]

\[ (x) \Delta x - (x \Delta x + x) \Delta x = \Delta x \]

\[ \Delta x = f(x) \]

The corresponding change in \((x)\) is approximated by the differential \(dx\).

A small change in \(x\) is denoted by the differential \(dx\).

\[ \int \Delta x = \int f(x) \Delta x \]

Let \( f \) be differentiable on an interval containing \( x \).

Definition
This means as \( x \) increases by 0.1, \( y \) decreases by 1.1.

\[
\frac{\Delta x}{\Delta y} = \frac{0.1}{1.1} = \frac{(x_2 - x_1)(0.1)}{1 - 3x_2} = hp
\]

For example, if \( x \) increases from 2 to 2.1, then \( dx = 0.1 \) and

\[
\Delta x = 0.1
\]

The change of \( p \) in \( y \) is

A small change in the variable produces an approximate change in the variable, so

\[
xp(x)(y_2 - y_1) = hp
\]

Solution: Given a small change \( dx \)

\[
[xp(x), f = hp]
\]

Use the notation of differentials to approximate the
3. (7 pts ea) Let $f(x) = \ln x - \sin(2 - x)$.

(a) Write the equation for the linear approximation to $f(x)$ at $x = 2$.

(b) Use your answer to (a) to approximate $f(1)$.

(c) Below is the graph of $f(x)$, drawn at the website desmos.com/calculator. On the same axis, draw your tangent line. Label both $f(1)$ and your approximation from part (b).
3. (7 pts ea) Let \( f(x) = \ln x + \sin(2-x) \).

(a) Write the equation for the linear approximation to \( f(x) \) at \( x = 2 \).

\[
\Delta y = f'(a)(x-a) = f(a) - f(2)
\]

\[
L(x) = f(a) + f'(a)(x-a)
\]

\[
L(x) = \ln 2 - \frac{1}{2} (x-2)
\]

\[
dy = L(a+\Delta x) - L(a)
\]

(b) Use your answer to (a) to approximate \( f(1) \).

\[
\Delta x = x - a = d x
\]

\[
f(1) \approx L(1) = \ln 2 - \frac{1}{2} (1-2) = \ln 2 + \frac{1}{2}
\]

\[
\Rightarrow dy = L(1) - \ln 2
\]

\[
= \frac{1}{2}
\]

(c) Below is the graph of \( f(x) \), drawn at the website desmos.com/calculator. On the same axis, draw your tangent line. Label both \( f(1) \) and your approximation from part (b).
1. (3 pts ea) Let \( g(x) = \ln(1 + x) \).

(a) Write the equation for the linear approximation to \( g(x) \) at \( x = 0 \).

(b) Use your answer to (a) to approximate \( g(0.9) \).

(c) Below is the graph of \( g(x) \), drawn at the website desmos.com/calculator. On the same axis, draw your tangent line. Label both \( g(0.9) \) and your approximation from part (b).