Exercise: Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and the graph of \( y = 8 - x^3 \).

Solution:
The graph \( y = 8 - x^3 \) is an upside-down cubic, shifted up by 8.

The width of the inscribed rectangle is just the x-coordinate, so call it \( x \). The height is the y-coordinate of the cubic at that given \( x \), so it's given by the formula \( 8 - x^3 \).

The objective function is the area of the rectangle, \( A = x(8 - x^3) \), and is already given in terms of one variable. The constraints also give the domain of \( A \): The smallest \( x \) can be is 0, since the y-axis is one of the boundary conditions.
the sides of the rectangle. The largest \( x \) can be is the \( x \)-intercept of the function \( y = 8 - x^3 \):

\[
0 = 8 - x^3 \implies x^3 = 8 \implies x = 2.
\]

So the domain is \([0, 2]\).

Differentiate:

\[
A'(x) = 8 - 4x^3 = 0
\]

\[
\implies x^3 = 2 \implies x = \sqrt[3]{2}.
\]

\( x = \sqrt[3]{2} \) is a critical point.

Verify the critical point gives a maximum by plugging it and the endpoints of the domain into the original area function:

\[
A(0) = (0)(8 - 0^3) = 0
\]

\[
A(\sqrt[3]{2}) = \sqrt[3]{2} \left( 8 - (\sqrt[3]{2})^3 \right) = \sqrt[3]{2}(8 - 2) = 6\sqrt[3]{2} \implies \text{max}.
\]

\[
A(2) = (2)(8 - 2^3) = 0.
\]

Now answer the question: The dimensions are \( \sqrt[3]{2} \times 6 \).

There are alternative ways to verify \( x = \sqrt[3]{2} \) gives a max.
1st Derivative Test:

\[ A'(0) = 8 - 4(0) = 8 > 0 \]
\[ A'(2) = 8 - 4(2^3) = 8 - 32 = -24 < 0 \]

\[ A' \] changes from \( \Theta \) to \( \Theta \) so \( x = \sqrt[3]{2} \) gives a local max;

Since this is the only local extremum on the interval \([0,2]\), it must be absolute.

2nd Derivative Test:

\[ A''(x) = -12x^2 < 0 \]

for all \( x \) means the area function is always concave down, and so the critical point must be a max; since it is the only local extremum on the interval \([0,2]\), it must be absolute.