1. Definitions

Examples will follow in Section 2. The most general setting is much broader than that taken here, but for the sake of establishing ideas, we will work in $\mathbb{R}^d$, real affine space, endowed with the usual metric and usual measure $\mu$; we let $G$ be any group of measure-preserving affine transformations on $\mathbb{R}^d$.

A tile is a compact, measurable set in $\mathbb{R}^d$. We restrict ourselves to using tiles with non-empty interior\(^1\). A configuration is a collection of tiles with disjoint interiors in $\mathbb{R}^d$. For a configuration $\tau$, the support $[\tau]$ of $\tau$ is defined as $[\tau] := \bigcup_{A \in \tau} A$. A tiling is a configuration with support $\mathbb{R}^d$. A species is a collection of tilings.

We adopt the convention, for any configurations $\tau_1, \tau_2$, that $\tau_1 \cup \tau_2$ is defined only if $\tau_1 \cup \tau_2$ is also a configuration (all tiles in $\tau_1 \cup \tau_2$ have disjoint interiors). Note that $\tau_1 \cap \tau_2 = \emptyset$ if and only if $[\tau_1] \cap [\tau_2] \subset \partial[\tau_1] \cap \partial[\tau_2]$.

Next, let $\sigma$ be an inflation, any expanding affine map with unique fixed point $o$, such that the cyclic group generated by $\sigma$ is normal in $<\sigma, G>$, the subgroup of all affine transformations from $\mathbb{R}^d$ to $\mathbb{R}^d$ generated by elements of $G$ and $\sigma$. We’ll call $o$ the origin\(^2\).

For all $g \in G$, for any integer $n$, there exists $h \in G$ with $\sigma^n g = h \sigma^n$. We define for each $g \in G$, a family $\{g^n \mid n \in \mathbb{Z}\}$ of $G$ such that $g^n g = g^n h \sigma^n$. It soon follows that, for all $n, m \in \mathbb{Z}$, $\sigma^m g^n = g^{n+m} \sigma^n$; $(g^n(m)) = g^{n+m}(m)$ and $(g^m)^n = (g^n)^m$.

Let $T$ be some finite set of tiles; the elements of $T$ will be called prototiles. With no loss of generality, we require that all the elements of $\{\sigma^n(A) \mid A \in T, n \in \mathbb{Z}\}$. For a tile $B = gA$, $g \in G$, $A \in T$, we will denote $B_T := A$, and $g_B := g$. Thus $B = g_B B_T$, $B_T \in T$. Let $C(T)$ be the set of configurations of the form $\{g_i A_i\}$, each $g_i \in G$, each $A_i \in T$.

A substitution is a map $\sigma' : C(T) \rightarrow C(T)$ such that for all $\tau \in C(T)$, $g \in G$,

(i) $\sigma'([\tau])$ is a configuration with $[\sigma'([\tau])] = \sigma([\tau])$.

(ii) $\sigma'(g\tau) = g^{(i)} \sigma'([\tau])$

(iii) For $\tau_0 \subset \tau$, $\sigma'(\tau_0) \subset \sigma'([\tau])$.

Note then that $\sigma'$ is determined by its action on the prototiles $T$, and that for $\tau_1, \tau_2 \in C(T)$, we have $\sigma'([\tau_1]) \cup \sigma'([\tau_2]) = \sigma'([\tau_1 \cup \tau_2])$.

Given $T$ and $\sigma'$, an $n$-level supertile is any configuration of the form $(\sigma')^n(A)$, or $(\sigma')^n(gA)$.

---

\(^1\) We need non-empty interior to ensure that $\Sigma(T, \sigma')$ is non-empty.

\(^2\) Note that the existence of such a $\sigma$ constrains $G$ quite a bit (In particular, no such $G$ exists if $G$ is the full group of measure preserving affine transformations).
Let \( A \in T, g \in G, n \in \mathbb{N} \). Note that our these are well defined by our conditions on \( \sigma' \).

The **substitution species** \( \Sigma(T, \sigma') \) of substitution tilings arising from \( T, \sigma' \) is the collection of tilings in \( C(T) \) such that \( \tau \in \Sigma(T, \sigma') \) if and only if for every \( \tau' \subseteq \tau \) with bounded support, there exists a supertile \( (\sigma')^n(gA) \) such that \( \tau' \subseteq (\sigma')^n(gA) \).

\( \Sigma(T, \sigma') \) has **unique decomposition** if and only if \( \sigma' : \Sigma(T, \sigma') \to \Sigma(T, \sigma') \) is one-to-one.

For each \( A \in T \), \( \sigma'(A) \) is a configuration; we will find it useful to denote the set of tiles in \( \sigma'(A) \) as \( A^+ \) and let \( S \) be the set of tiles given by

\[
S := \bigcup_{A \in T} A^+ \tag{3}
\]

Note that \( S \) is partitioned into the \( A^+ \) (this is the sole reason we required that the elements of \( T \) be disjoint).

For all \( A \in S \), denote \( A^+ := (A_T)^+ \)

For all \( A \in S \), there is a unique \( B \in T \) with \( A \in B^+ \). Let \( A^- := B \)

For all \( A \in T \), let \( A^- := \{B^- \in T \mid B \in S, B_T = A \} \).

Note \( S \) is finite (since the elements of \( T \) have non-zero measure and so each \( \sigma'(A), A \in T \)

consists of finitely many tiles). Index the elements of \( S \) so that \( S = \{S_i\} \) and define the \( |S| \times |S| \) matrix \( S \) by

\[
S_{ij} := \begin{cases} 1 & \text{if } S_j \in (S_i)^+ \\ 0 & \text{otherwise} \end{cases}
\]

Define **addresses** \( A = \Sigma_S \subset S^Z \), the two sided subshift of finite type, with alphabet \( S \) defined by the matrix \( S \). It will also be convenient to associate a **substitution graph** \( \Gamma(T, \sigma') \) with \( A \); this graph will have vertices indexed by \( T \) and directed edges indexed by \( S \); an edge, indexed \( A \), leaves a vertex \( X \) for vertex \( Y \) if and only if \( Y = A_T, A \in T^+ \). Thus the elements of \( A \) exactly correspond to bi-infinite paths of edges in \( \Gamma(T, \sigma') \).

Let \( \varsigma \) be the **shift** acting on \( \Sigma_S \) by \( \varsigma(\alpha) = \beta \) iff \( \alpha_n = \beta_{n+1} \) for all \( n \in Z \).

We will take **blocks** \( [A_n \ldots A_m] := \{\alpha \in A \mid \alpha_j = A_j \text{ for } n \geq j \geq m\} \) as a basis for a topology on \( A \). For each \( \alpha \in A \) it will be useful to abuse notation and define \( [\alpha] := \{\beta \in A \mid \exists n \in Z \text{ s.t. } \forall k \geq n, \beta_k = \alpha_k\} \). Note \( A \) is partitioned into these \( [\alpha] \).

For all \( n \in Z \), let \( A^n = \{[A_n \ldots]\} \), the set of all infinite-to-the-right blocks with first index \( n \). Let \( \varsigma : A^n \to A^{n+1} \) in the obvious manner.

For any block \( [A_n \ldots A_m] \subset A \), define \( g_{[A_n \ldots A_m]} := g_{A_n}^{(n)} \ldots g_{A_j}^{(j)} \). Now define the map \( \lambda : A^n \to \mathbb{R}^d \) by

\[
\lambda([A_n \ldots]) := \bigcap_{j \leq n} g_{[A_n \ldots A_j]} \sigma^j(A_j)_T \tag{4}
\]

Now we should note the following:

**Lemma 1.1**

1. \( \lambda : A^n \to \mathbb{R}^d \) is well defined.

2. \( \lambda(\varsigma([A_n \ldots])) = \sigma\lambda([A_n \ldots]) \).

3. \( \lambda([A_n]) = \sigma^n(A_n) \) (Note \( [A_n] = \bigcup_{[B_n \ldots] \in A^n, A_n = B_n} [B_n \ldots] \))

\( ^3 \)That is, \( A^+ := \sigma'(A) \), but we regard \( A^+ \) as a set of symbols and \( \sigma'(A) \) as a configuration of tiles.

\( ^4 \)denoted by \( \lambda \) since we are Aabling points.
4. \( \lambda([A_n \ldots A_0]) = g_{[A_n \ldots A_0]}(A_n)T = g_{[A_n \ldots A_1]}A_0 \)

5. For \( n \in \mathbb{N}, A \in S \), the supertile \((\sigma')^n(A) = \{ \lambda([A_n \ldots A_0]) \mid A_n = A \}\)

Proof (1) That \( \lambda([A_n \ldots ]) \) is a single, well-defined point in \( \mathbb{R}^d \) follows from the observation that the sets \( g_{[A_n \ldots A_j]}(A_j)T \) are nested, closed, and have diameters going to 0 as \( j \) goes to \(-\infty\). So there is a single point in the intersection of these sets.

(2-5) follow from the definitions. Note that (3) is a statement about points in \( \mathbb{R}^d \), and (5) is a statement about configurations in \( \mathbb{R}^d \). We point out (4) to help keep in mind both the meaning of \( \lambda \) and the meaning of \( g_{[A_n \ldots A_0]} \). \( g_{[A_n \ldots A_0]} \) gives the position of \((A_0)T \) in \((\sigma')^n(A_n)\).

We now define a pair of relations: \( \approx \) is an equivalence relation on \( \mathbb{A}^n \); \(| \) is a reflexive, symmetric relation on blocks in \( \mathbb{A}^n \):

For \( \alpha, \beta \in \mathbb{A}^n \), \( \alpha \approx \beta \) if and only if \( \lambda(\alpha) = \lambda(\beta) \).

For blocks \( A, B \subset \mathbb{A}^n \), \( A \neq B \) if and only if there exist \( \alpha \in A, \beta \in B \) with \( \alpha \approx \beta \). Note that for blocks \( A \neq B \) if and only if \( \lambda(A) \subset \lambda(B) \) or \( \lambda(B) \subset \lambda(A) \) or \( \lambda(A) \cap \lambda(B) \neq \emptyset \). We can now define \( \mathbb{A}^n \) as configurations of the form \( \{ [\alpha_k \ldots] | \beta_k \ldots \} \in \mathbb{A}^k \) with \( [\alpha_k \ldots] \neq [\beta_k \ldots] \), \( g = g_{[\alpha_k \ldots \alpha_{k+1}]} \), \( h = g_{[\beta_k \ldots \beta_{k+1}]} \), \( A = \alpha_n, B = \beta_n \) and finally \( (\sigma')^n(A) = \{ \lambda([\alpha_n \ldots A_0]) \} \), \( (\sigma')^n(B) = \{ \lambda([\beta_n \ldots B_0]) \} \).

This can be interpreted as:

**Lemma 1.2** Let \( A, B \in S, g, h \in G, n \in N \). Then \( g(\sigma')^n(A)|h(\sigma')^n(B) \) if and only if there exists some \( k \in \mathbb{N}, k > n, X \in T \) with \( g^{(n-k-1)}A, h^{(n-k-1)} \) adjacent tiles in \( (\sigma')^{k+1-n}(X) \).

This is not hard to verify and proof is omitted.

The definition of \( \lambda([A_n \ldots ]) \) was “top-down”. In particular \( \lambda([A_n \ldots A_0]) \) gives the position of a tile within a fixed supertile \((\sigma')^n((A_0)T) \). To define a map from \( \mathbb{A} \) we will need a “bottom-up” construction that fixes the position of a configuration given the position of a tile. To do this we define a host of maps:

For each \( \alpha \in \mathbb{A} \), define \( \lambda_\alpha : [\alpha] \to \mathbb{R}^d \) as follows. Given \( \beta \in [\alpha] \), let \( n \) be any integer such that for all \( k \geq n \), \( \alpha_k = \beta_k \).

\[ \lambda_\alpha(\beta) := (g_{[A_n \ldots]}^{-1})^{-1} \lambda([\beta_n \ldots]) \]

Note that this does not depend on \( n \) we use, and that \( \lambda_\alpha(\alpha) = \varnothing \). One ought verify that \( g_{[A_n \ldots]} := \lim_{j \to -\infty} (g_{[A_n \ldots A_j]}^{-1}) \) is a well-defined affine transformation in \( G \).

We can now define **infinite level supertiles** as configurations of the form \( \tau_\alpha := \{ \lambda_\alpha([\ldots A_0] \mid [\beta] \in [\alpha]) \} \), and of the form \( g\tau_\alpha, g \in G \).

Note for any infinite level supertile \( g\tau_\beta \), that \( o \in g\tau_\beta \) if and only if there is an \( \alpha \in [\beta] \) such that \( g\tau_\beta = \tau_\alpha \).
We define \( | \) on infinite-to-the-left-blocks as follows. Let \( \alpha, \beta \in \mathcal{A} \); then \( \alpha | \beta \) if and only if there exist: \( \alpha' \in [\alpha], \beta' \in [\beta] \), such that for all \( m \in \mathbb{Z} \), there is an \( n \in \mathbb{Z}, n > m \), and \( \gamma \in \mathcal{A} \) such that \( \alpha'_{m-1}, \beta'_{m-1} \in (\gamma_m)^+ \) and \( [\gamma_n \cdots \gamma_m \alpha'_{m-1}] [\gamma_n \cdots \gamma_m \beta'_{m-1}] \).

This can be interpreted as follows (we skip the proof):

**Lemma 1.3** If \( \tau \in \Sigma(T', \sigma) \) contains infinite-level supertiles \( g\tau_\alpha, h\tau_\beta \) such that \( [g\tau_\alpha] \cap [g\tau_\beta] \neq \emptyset = \text{int} [g\tau_\alpha] \cap \text{int} [g\tau_\beta] \), then \( \alpha | \beta \)

Accordingly, in an abuse of notation we define \( | \) on infinite level supertiles as \( g\tau_\alpha | h\tau_\beta \) if and only if \( [\alpha] [\beta] \) or \( [g\tau_\alpha] \cap [h\tau_\beta] = \emptyset \).

### 2. Examples

We would be well served by examples before continuing.

**Example 2.1**

Our first example is the \( L \)-tiling or “chair” tiling in \( \mathbb{R}^2 \). \( T \) consists of one tile, \( L \); \( \sigma \) doubles all distances; \( \sigma' \) expands each \( L \)-tile and replaces each \( \sigma L \) with four \( L \)-tiles. \( L, \sigma' L, \) and \((\sigma')^2 L \) are shown in figure 1(i). In figure 1(ii) is a portion of what appears to be an element of \( \Sigma(\{L\}, \sigma') \).

In figure 1(iii) we have illustrated \( S = L^+ = \{a, b, c, d\} \). \( \mathcal{K} \) consists of all bi-infinite strings of these letters; \( \Gamma(\{L\}, \sigma') \) is shown in (iv).

In figure 2 we have illustrated some things connected to our first map \( \lambda \). Recall that \( S \) can be regarded as both a set of tiles and as a set of letters. The elements of \( S \) are in boldface when used to indicate tiles.

In figure 3 we have illustrated some infinite level supertiles. Note the location of \( \sigma! \)

It can be noted that the \( L \)-substitution species has unique decomposition—every \( L \)-substitution tiling arises in exactly one way as the image of another \( L \)-substitution tiling under the map \( \sigma' \). (That is, one can uniquely cluster together the \( L \)-tiles into 1-level supertiles.)

![Figure 1: The L substitution tiling](image1.png)

**Example 2.2** Our second example is quite familiar; \( T \) consists of one tile, a segment in \( \mathbb{R}^1 \). \( \sigma' \) simply doubles the length of the segment and replaces it with two segments. \( S \) will be denoted

![Figure 1: The L substitution tiling](image2.png)
Figure 2: $\lambda$ and the $L$ substitution tiling

Figure 3: Two infinite level supertiles in the $L$ substitution tiling
and $A$ consists of all bi-infinite strings of these digits. One special infinite supertile, $\tau_0$ can be regarded as the binary representation of the non-negative real numbers. Note for example that $\overline{100.1101001}$ and that $[100.11][101]$. This example does not have unique decomposition— all tilings in the species are equivalent up to translation and $\sigma'$ is two-to-one.

**Example 2.3** Here $T$ consists of the three triangles shown; $\sigma$ is a dilation of magnitude $s \approx 1.324717957244746$, the real root of $s^3 - s - 1 = 0$. $\sigma'$ takes the elements of $T$ to supertiles as shown. A small portion of a substitution tiling in $\Sigma(T, \sigma')$ is at right in (iii).

$S$ and the substitution graph are illustrated in figure 4(ii).

![Figure 4: A substitution on triangles](image)

### 3. Beginning Results

For the following, fix $G$, $T$ and $\sigma$ and $\sigma'$ in $\mathbb{R}^d$. Then $S$ and $\lambda$ are determined.

**Theorem 3.1**

1. Infinite level supertiles are well defined, unbounded configurations in $C(T)$.

2. For $\alpha, \beta \in A$, if $\beta \in [\alpha]$ then there is a $g \in G$ with $g \tau_\alpha = \tau_\beta$. If $\Sigma(T, \sigma')$ has unique-decomposition then the converse holds as well. Also, $\tau_{\alpha\beta} = \sigma' \tau_\alpha$.

This essentially follows from the definitions and is not hard to verify. Note that in fact, we can explicitly give the $g$ in (2) above: if $\beta \in [\alpha]$, there exists an $n \in \mathbb{N}$ such that for all $k > n$, $\alpha_k = \beta_k$. Then for $g = g_{[\beta_k,...]}(g_{[\alpha_n,...]})^{-1}$, we have $g \tau_\alpha = \tau_\beta$.

The second and third parts of the following are well known “folk-theorems”. The statements strongly depend on the tiles in $T$ having non-empty interior.

**Theorem 3.2**

1. There is an infinite level supertile that is a tiling in $\Sigma(T, \sigma')$.

2. Thus, $\Sigma(T, \sigma')$ is non-empty.

3. In fact, if $\Sigma(T, \sigma')$ has unique decomposition, then $\Sigma(T, \sigma')$ has uncountably many non-congruent tilings.
Proof  (1) Since \(\sigma(A)\) has greater measure than \(A\), for each element of \(T\), it follows that \(|S| > |T|\). We claim that there is an \(\alpha \in \mathbb{A}\) satisfying:

(i) There is an \(n \in \mathbb{N}\) with \(\zeta^n\alpha = \alpha\) and

(ii) \(\lambda(\alpha_0, \alpha_{-1} \ldots) \in \text{int}\lambda([\alpha_0])\)

We construct the desired \(\alpha\) as follows: First note that there must be some \(A \in S\) with \(gA \in \text{int}((\sigma')^n(A))\) for some \(g \in G, n \in \mathbb{N}\). This must be so since \(\sigma\) expands all distances and each tile is bounded; thus the number of tiles strictly in the interior of each \((\sigma')^n(A)\), \(A \in T\) must grow without bound as \(n\) grows. Since \(T\) is finite, we must have that there exists some \(A \in T\), \(g \in G, n \in \mathbb{N}\) with \(gA \subset \text{int}(\sigma')^nA\).

Now this \(g = g_{[A_n, A_0]}\) for some block \([A_n, A_0] \subset \mathbb{A}\) with \(A_n = A_0 = A\). Take \(\alpha \in \mathbb{A}\) to be such that \(\alpha_i = A_k\) where \(i \equiv k\mod (n - 1)\). Then this \(\alpha\) satisfies our claim

Now it is not difficult to see that \(\lambda_n([\alpha]) \in \Sigma(T, \sigma')\) in other words, \([\lambda_n([\alpha])]) = \mathbb{R}^d\).

(2) Thus \(\Sigma(T, \sigma')\) is not empty.

(3) Now we show that \(\Sigma(T, \sigma')\) is uncountable if \(\Sigma(T, \sigma')\) has unique decomposition. From the proof of 1 we can quickly see that there is an \(A \in S\), \(n \in \mathbb{N}\) such that there are two distinct images of \(A\) in the interior of \((\sigma')^n(A)\). Let us denote these images \(B = \lambda([B_n, \ldots, B_0])\) and \(C = \lambda([C_n, \ldots, C_0])\), \(B_n = B_0 = C_n = C_0 = A\). Now define \(f : \{0, 1\}^2 \to \Sigma(T, \sigma')\) by:

\[
f(s) = \tau_{\alpha}\text{ where, for } j, k \in \mathbb{Z}, 0 \leq k < n, \alpha_{jn+k} = B_k \text{ if } s_j = 0, \alpha_{jn+k} = C_k \text{ if } s_j = 1.
\]

As before, it is not hard to verify that each such \(\tau_{\alpha}\) has support \([\tau_{\alpha}] = \mathbb{R}^d\) and thus is a tiling in \(\Sigma(T, \sigma')\).

We next claim that if \(\Sigma(T, \sigma')\) has unique decomposition, none of the images of \(f\) can be congruent to one another. This follows from Theorem 3.1,(2) since for all \(\alpha, \beta \in f(\{0, 1\}^2)\), \(\alpha \notin [\beta]\). Moreover \(f\) is one-to-one, and so 3 is proved.

The following theorems show that these definitions really capture the structure of \(\Sigma(T, \sigma')\).

Proposition 3.3 The elements of \(\Sigma(T, \sigma')\) are precisely of the form \(\bigcup g_i \tau_{\alpha_i}\) where \(\{\alpha_i\}\) is a countable subset of \(\mathbb{A}\) with \(g_i \tau_{\alpha_i}, g_j \tau_{\alpha_j}\) for each distinct pair \(i, j\).

We would like to go on to add that in fact \(\cap [g_i \tau_{\alpha_i}]\) consists of a single point, and that there can be only finitely many such \(g_i \tau_{\alpha_i}\); we can’t see a proof though, and perhaps this is not true for all \(\Sigma(T, \sigma')\).

Proof Before we begin, we show that for any point \(x \in \mathbb{R}^d\), for any \(n \in \mathbb{N}\), there is a supertile \(g(\sigma')^n(A) \subset \tau\) with \(x \in \sigma^n(A)\). To see this, let \(d\) be the maximum diameter of any tile in \(T\), and let \(D\) be a ball of radius \(2d\) with center \(w^{-n}(x)\). Then consider the configuration \(\tau'\) of tiles in \(\tau\) with support intersecting \(w^kD\). Then (by the definition of \(\Sigma(T, \sigma')\)) there is a supertile \(g(\sigma')^k(B) \supset \tau'\). Now we must have \(k > n\) and that \(g(\sigma')^k(B)\) is the disjoint union of supertiles of the form \(\lambda([\beta_k, \ldots, \beta_n])\) with \(\beta_k = B\). One of these must have support containing \(x\), and is therefore contained in \(\tau'\) and hence in \(\tau\).

We now show that for all \(x \in \mathbb{R}^d\) there is an infinite level supertile \(g \tau_{\alpha} \subset \tau\) with \(x \in \lfloor g \tau_{\alpha}\rfloor\). It suffices to show that any \(\tau \in \Sigma(T, \sigma')\) there is an infinite-level supertile \(\tau_{\alpha} \subset \tau\) with \(x \in \lfloor \tau_{\alpha}\rfloor\).

Let \(S_n\) be the collection of all \(n\)-level supertiles in \(\tau\) with support containing \(\alpha\). If \(h(w')^n(A) \in S_n\), must be a \(B \in A^+\) with \(h \gamma(w')^n(B) \in S_{n}\). Now since no \(S_n\) is empty, there must be, in fact, a sequence of \(A_i, i = 0, 1, \ldots\) such that there is a \(h_i \in G\) with \(h_i(w')^n(A_i) \in S_n, \text{ and } A_{i-1} \in A_i^+\) with \(h_i \gamma(h_i) = h_{i-1}\). Now there exists an \([B_0, \ldots] \in \mathbb{A}^n\) such
that \( A_0 = B_0 \) and \((g_{|B_0...})^{-1}(B_0)_T = A_0 \). Then it is not hard to verify that for \( \alpha \in \Lambda \) with \( \alpha = \ldots A_n \ldots A_0, B_{-1} \ldots \), \( \tau_\alpha \subset \tau \) with \( \alpha \in [\tau_\alpha] \).

We now claim there is a collection of infinite-level supertiles in \( \tau \) with support \( \mathbb{R}^d \). We first define a collection \( \{g_i\tau_\alpha\}_{1 \leq i \leq n} \) of infinite level-supertiles to be ok if and only if each \( g_i\tau_\alpha \subset \tau \) and for all distinct \( i, j, g_i\tau_\alpha, g_j\tau_\alpha \).

We prove the following statement by induction on \( n \): If \( \{g_i\tau_\alpha\}_{1 \leq i \leq n} \) is a finite o.k. collection of infinite-level supertiles then either \( \tau = \bigcup g_i\tau_\alpha \) or \( \{g_i\tau_\alpha\}_{1 \leq i \leq n+1} \) such that \( \{g_i\tau_\alpha\}_{1 \leq i \leq n+1} \) is o.k.

If \( \tau = \bigcup g_i\tau_\alpha \) then we are done; otherwise there is an \( x \in \mathbb{R}^d \) not in the interior of the support of any of the \( g_i\tau_\alpha \). With no loss of generality, we may assume \( x = 0 \) and that \( 0 \) is on the boundary of one or more of the \( g_i\tau_\alpha \). Essentially by repeating the argument at the beginning of the proof we can construct the needed \( g_n+1\tau_{\alpha_{n+1}} \) such that \( \{g_i\tau_\alpha\}_{1 \leq i \leq n+1} \) is o.k. \( \square \)

The following is a well-known “folk theorem”. In a way, this idea is already present in [ ] (even though he is not, strictly speaking, considering substitution tilings). We proved this within the proof of Proposition 3.3.

**Theorem 3.4** For any \( \tau \in \Sigma(T, \sigma') \), for any tile \( A \) in \( \tau \):

1. for any \( n \in \mathbb{N} \), there is a \( g \in G, B \in T \) such that the \( n \)-level supertile \( g(\sigma')^n(B) \) is a subset of \( \tau \) and \( A \in g(\sigma')^n(B) \).

2. There is an \( g \in G, \alpha \in \Lambda \) such that the infinite level supertile \( g\tau_\alpha \) contains \( A \).

This provides a kind of converse to Proposition 3.3.

**Theorem 3.5** If \( S \) is irreducible, then every infinite level supertile is a subset of some element of \( \Sigma(T, \sigma') \). In fact, if \( g \ldots *** \)

It is not hard to cook up examples in which \( mS \) is not irreducible and there is a finite level supertile that does not appear in any element of \( \Sigma(T, \sigma') \). Consider the following example: \( M = \mathbb{R}^2, G \) consists only of translations. There are two tiles in \( T \), colored squares, with the substitutions shown. The infinite-level supertile pictured at right is not in any tiling in \( \Sigma(T, \sigma') \).

**Proof of Theorem 3.5:** One can verify that if \( S \) is irreducible, then for every pair \( A, B \in S \) there exists a \( g \in G, n \in \mathbb{N} \) with \( gA \subset \text{int}(w^n(B)) \).

Now let \( \tau_\alpha, \alpha \in \Lambda \) be any infinite level supertile. If \( [tau_\alpha] = \mathbb{R}^d \) then \( \tau_\alpha \in \Sigma(T, \sigma') \) and we are done. So assume \( [\tau_\alpha] \neq \mathbb{R}^d \).

Choose any tile \( h'A, h' \in G, A \in T \) on the boundary of \( \tau_\alpha, \tau_\alpha \). We may assume, after possibly applying an isometry \( g \) and relabeling \( \alpha \), that \( h \) is the trivial isometry (the identity of \( G \), and that \( \alpha_0 = A \).
Let \( D_0 \) be a ball centered at \( \circ \) with radius \( 2d \), where \( d \) is the maximum diameter of the tiles in \( T \), and let \( D_k = \sigma^k D_0 \).

Let \( \Sigma_k \) consist of all configurations \( \tau' \) in \( C(T) \) such that

(i) \( \tau' \cap D_k \);
(ii) there exists a supertile \( g(\sigma')^n(B) \cap \tau_k \); and
(iii) \( \tau_k \cap \{ A \in \tau_n \mid \{ A \} \cap D_n \neq \emptyset \} \).

We will show that there exists a sequence of \( \tau_k \in \Sigma_k \), with \( \tau_k \subset \tau_{k+1} \). The union of the \( \tau_k \) in this sequence will be a well-defined tiling in \( \Sigma(T, \sigma') \), containing \( \tau_\alpha \).

Claim: each \( \Sigma_k \) is non-empty.

Proof of claim: By the definition of \( \tau_\alpha \), there must exist an \( n \) such that \( \lambda'_h(A_n \ldots A_0) \supset D_k \cap \lambda'_h(\ldots A_0) \). Now there exists an \( m \in \mathbb{N} \), \( h'' \in G \), \( B \in T \) such that \( h''(A_n) \tau \) is in the interior of \( (\sigma')^m(B) \).

Note that \( (h''(n)) (\sigma')^n(A_n) \subset (\sigma')^{m+n} \) and that for \( \alpha H \in (h''(n)) (\sigma')^n(A_n) = \lambda'_h(A_n \ldots A_n) \),
\[
H = h \ (g_{A_n \ldots A_n})^{-1} (h''(n))^{-1}
\]

Now choosing \( m \) suitably large, we can assume that
\[
H^{-1} D_k \subset \sigma^{m+n}(B)
\]

Define then \( H^{-1} \Sigma_k \) to be \( (h''(n)) (\sigma')^n(A_n) \cup ((\sigma')^{m+n} \cap H^{-1}) D_k \).

The claim is proved.

Note that by the definition of each \( \Sigma_k \), if \( \tau, \tau' \in \Sigma_k \), \( \tau \cap D_k \subset \tau_{k-1} \); thus there is a sequence \( \{ \tau_k \in \Sigma_k \} \), \( \tau_k \subset \tau_{k+1} \), and \( \cup \tau_k \) is a tiling in \( \Sigma(T, \sigma') \) containing \( \tau_\alpha \).

4. Algorithmic descriptions of orbits in \( \Sigma(T, \sigma') \) under \( \sigma' \)

**Theorem 4.1** \( \sigma' : \Sigma(T, \sigma') \to \Sigma(T, \sigma') \) is onto.

**Proposition 4.2** If there are finitely many local configurations in \( \Sigma(T, \sigma') \), then there is a Mealy machine that precisely determines the relation \( | \) on \( \mathcal{A} \).

**Corollary 4.3** There is an algorithm for determining the finite orbits of \( \sigma' \) acting on \( \Sigma(T, \sigma') \). If \( \Sigma(T, \sigma') \) has finitely many local configurations there is an algorithm for determining all orbits of \( \sigma' \) acting on \( \Sigma(T, \sigma') \).

**Corollary 4.4** Given \( \Sigma(T, \sigma') \) there are countably many self-similar tilings in \( \Sigma(T, \sigma') \), finitely many of each inflation factor.

The following is the result of a calculation based on Corollary 4.3. Of course, infinitely many similar sounding corollaries are also available!

**Corollary 4.5** Under the action of \( \sigma' \), the \( L \) substitution species has precisely four orbits of length 1, six orbits of length 2 and sixteen orbits of length three.

**Theorem 4.6** If \( \Sigma(T, \sigma') \), \( \Sigma(T_0, \sigma'_0) \) are well enough behaved (polyhedral tiles, her. facets and sibling \( v \omega v \)) and have homomorphic addressings, then in some fundamental they are the same tiling. \ldots