

# A PROOF OF WALDHAUSEN'S UNIQUENESS OF SPLITTINGS OF $S^3$ (AFTER RUBINSTEIN AND SCHARLEMANN)

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In [6] Waldhausen proved uniqueness of non-stabilized Heegaard splittings of  $S^3$ :<sup>1</sup>

**Theorem 1** (Waldhausen). *Let  $\Sigma \subset S^3$  be a Heegaard surface of genus  $g > 0$ . Then  $\Sigma$  is a stabilization of a Heegaard surface of genus  $g - 1$ .*

In [3] J.H. Rubinstein and M. Scharlemann used Cerf Theory to compare Heegaard splittings of irreducible, non-Haken manifolds. As a corollary of their work they obtained a new proof of Theorem 1. The purpose of this note is developing Cerf Theory directly in  $S^3$ . This allows us to use Rubinstein and Scharlemann's philosophy and obtain a simpler proof of theorem 1.

We begin with an outline of the proof. As with many proofs of theorem 1 we assume the theorem is false and pick  $\Sigma$  to be a minimal genus counter example; we induct on  $g$ , the genus of  $\Sigma$ . A simple application of van-Kampen's theorem shows that if  $g = 1$  the meridians of the complementary solid tori intersect minimally once and hence  $\Sigma$  is a stabilization of the genus zero splitting of  $S^3$ . The heart of the argument (in the following sections) shows that  $\Sigma$  weakly reduces. By A. Casson and C. McA. Gordon's seminal work [1] either  $\Sigma$  reduces or  $S^3$  contains an essential surface (*i.e.*,  $S^3$  is reducible or Haken). As the latter is impossible,  $\Sigma$  must reduce. Cutting  $S^3$  open along the reducing sphere we obtain 2 balls (say  $B_1$  and  $B_2$ , resp.) and a once punctured surface in each (say  $S_1$  and  $S_2$ , resp). We attach 3-balls to  $B_1$  and  $B_2$  and cap off  $S_1$  and  $S_2$  with disks. It is easy to see that we obtain two Heegaard splittings of  $S^3$ , and each has genus less than  $g$  and positive. By our inductive hypothesis each of them is stabilized. Hence,  $\Sigma$  is stabilized as well.

The remainder of this article is devoted to showing that  $\Sigma$  weakly reduces.

## 1. THE GRAPHIC

$S^3$  is the unit ball in  $\mathbb{R}^4$ . As such, it inherits a height function given by the projection onto the  $z$ -axis, denoted  $h_1$ ; then  $S^3$  has one maximum at  $(0, 0, 0, 1)$ , one minimum at  $(0, 0, 0, -1)$  and for any  $s \in (-1, 1)$  we have that  $h_1^{-1}(s)$  is a 2-sphere which we denote  $S_s^2$ . This is a special case of a *sweepout*.

Given  $\Sigma$ , we have a sweepout of  $S^3$  corresponding to  $\Sigma$ ; we briefly describe it here and refer the reader to [3] for details. Let  $S^3 = U \cup_{\Sigma} V$  be the Heegaard splitting corresponding to  $\Sigma$ . Let  $h_2$  be a height function on  $U$ ,  $h_2 : U \rightarrow [-1, 0]$  so that  $\partial U = \Sigma$  is at level 0, a spine of  $U$  is at the level -1 and for each  $t \in (-1, 0]$ ,  $h_2^{-1}(t)$  is a surface parallel to  $\partial U$ . Similarly take a height function on  $V$  (also denoted  $h_2$ )  $h_2 : V \rightarrow [0, 1]$ , so that  $\partial V = \Sigma$  is at level 0, a spine of  $V$  is at the level 1 and

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*Date:* June 15, 2005.

*1991 Mathematics Subject Classification.* 57M99, 57M25.

*Key words and phrases.* Heegaard splittings, 3-sphere, Poincarè conjecture, Cerf theory.

This work was done while I was visiting Nara Women's University. I thank Tsuyoshi Kobayashi for his kind hospitality.

<sup>1</sup>We assume familiarity with the basic facts and standard terminology of 3-manifold topology and in particular Heegaard splittings; see [4].



FIGURE 1. Obtaining the Board

for each  $t \in [0, 1)$ ,  $h_2^{-1}(t)$  is a surface parallel to  $\partial V$ . Pasting the two functions together and obtain a function  $h_2 : S^3 \rightarrow [-1, 1]$ . For  $t \in (-1, 1)$  we denote  $h_2^{-1}(t)$  by  $\Sigma_t$ .<sup>2</sup> We assume that the spines of  $U$  and  $V$  are disjoint from  $(0, 0, 0, 1)$  and  $(0, 0, 0, -1)$ . For every point  $(s, t) \in (-1, 1) \times (-1, 1)$  we have the two surfaces  $S_s^2$  and  $\Sigma_t$ . Cerf Theory says that we can perturb  $h_1$  and  $h_2$  so that the intersection of  $S_s^2$  and  $\Sigma_t$  is transverse for almost all  $(s, t) \in [-1, 1] \times [-1, 1]$ , and the set for which the intersection is not transverse form a finite graph (called the *Graphic*) with the following properties:

- (1) For  $(s, t)$  on an edge of the graphic,  $S_s^2 \cap \Sigma_t$  contains exactly one non-degenerate critical point (either center or a saddle).
- (2) At a valence 4 vertex the corresponding surfaces with exactly two non-degenerate critical points. A valence 4 vertex can be seen as a point where two arcs of the graphic that cross each other, each corresponding to a single non-degenerate critical point.
- (3) There is one other type of vertex (called *Birth-Death* vertex) that has valance 2. Birth-death vertices do not play a role in our study and we will not describe them here.

The closure of a component of  $[-1, 1] \times [-1, 1]$  cut open along the Graphic is called a *region*. Given a region, the intersection of the surfaces that correspond to a point in the region does not depend in the choice of point in any essential way.

## 2. THE LABELS $I$ , $E$ , AND A FRIENDLY GAME OF HEX

We label the regions. The label  $E$  (stands for “essential”) is used whenever the intersection of surfaces corresponding to a point on the region contain a curve that is essential in  $\Sigma_t$ ; otherwise, the label  $I$  (stands for “inessential”) is used. By definition each region has exactly one label.

In order to enjoy a game of Hex we modify the Graphic as follow: if a valance 4 vertex is adjacent to two  $E$ -regions and two  $I$ -regions and the labels alternate when going cyclically around it, we split the Graphic and introduce a short edge separating the  $I$  regions; see Figure 1 where the northern and southern are  $I$ -regions and the western and eastern regions are  $E$ -regions. The graph obtained is called the *Board*. Note that there is a natural correspondence between regions of the Graphic and those of the Board; using this correspondence the regions of the Board inherit labels from the Graphic.

The reason for creating the Board is:

**Proposition/Definition 2.1.** *By the border we mean the union of the edges of the Board that separate  $E$ -regions from  $I$ -regions .*

*In  $(-1, 1) \times (-1, 1)$  the border forms an embedded 1-manifold.*

*Proof.* Away from the vertices the proposition clearly holds. Let  $v$  be a valance 4 vertex. If all regions around  $v$  have the same label  $v$  isn't on the border. If one region has one label and three have the other label, the border is locally an interval (with a corner). If two regions adjacent to  $v$  are labelled  $E$  and two are labelled

<sup>2</sup>This description of sweepout is equivalent to the description given in [3] but has a different appearance; for a detailed description of this approach see [2].

$I$ , then by construction of the Board the label do not alternate. Hence the Border cuts across such vertex smoothly. Again by construction of the Board, at a valance 3 vertex a single  $E$ -region meets two  $I$ -regions, and the Border simply follows the boundary of the  $E$ -region. (Clearly, a valance 2 vertex cannot pose a problem.)  $\square$

We now pick two volunteers to play Hex. The goal of the first volunteer, Ivan, is finding a chain of regions (say  $R_1, \dots, R_n$ ,  $n \geq 1$ ) labelled  $I$  that connects the left edge of the Board (points with  $s = -1$ ) with its right ( $s = 1$ ). Similarly, the goal of the second volunteer, Esmeralda, is finding a chain of regions labelled  $E$  that connects the bottom edge of the board ( $t = -1$ ) with its top ( $t = 1$ ). For  $i = 2, \dots, n$  the region  $R_{i-1}$  shares an edge with  $R_i$ .

The next proposition is quite special to  $S^3$ :

**Proposition 2.2.** *Ivan can't win.*

*Proof.* Suppose Ivan wins and let  $R_1, \dots, R_n$  (for some  $n$ ) be a chain of regions, starting at the left ( $s = -1$ ) and ending at the right<sup>3</sup> ( $s = 1$ ) with  $R_{i-1}$  and  $R_i$  sharing an edge (for  $i = 2, \dots, n$ ). Consider the corresponding regions in the Graphic (still denoted  $R_i$ ). The cost: since some edges of the Board are crushed, it is now possible that  $R_{i-1}$  shares only a valance 4 vertex with  $R_i$ . Given  $s \in [-1, 1]$  we color  $h_1^{-1}([-1, s])$  yellow and  $h_1^{-1}([s, 1])$  green.

The following lemma is an easy application of innermost disk argument and its proof is left for the reader:

**Lemma/Definition 2.3** (regarding  $I$ -regions). *If  $(s, t)$  is in an  $I$ -region then the entire surface  $\Sigma_t$  (except perhaps for parts contained in a disk) is either yellow or green (resp.); we say that  $\Sigma_t$  is essentially yellow (green resp.).*

We replace the labels  $I$  by labels  $I(G)$  and  $I(Y)$  as follows:  $I$ -regions with essentially green surfaces is labelled  $I(G)$  and  $I$ -regions with essentially yellow surfaces is labelled  $I(Y)$ . Of course no surface is essentially green and essentially yellow simultaneously; this, together with Lemma 2.3, establishes that every  $I$ -region gets exactly one label. In addition, it is easy to see that  $I$ -regions with  $s$  very close to -1 are labelled  $I(G)$  and  $I$ -regions with  $s$  very close to 1 are labelled  $I(Y)$ . Considering the chain of  $I$ -regions  $R_1, \dots, R_n$ , we see that  $R_1$  is labelled  $I(G)$  and  $R_n$  is labelled  $I(Y)$ . Let  $i$  be the first index with  $R_i$  labelled  $I(Y)$ . Thus  $R_{i-1}$  is labelled  $I(G)$  and  $R_i$  is labelled  $I(Y)$ . If  $R_{i-1}$  and  $R_i$  share an edge then passing from one to the other we cross a single critical point, either a center or a saddle. In either case, no essential curve is introduced or removed (recall we are crossing from one  $I$ -region to another) and therefore labels cannot change. Thus we may assume that we cross a valance 4 vertex (say  $v$ )<sup>4</sup>, corresponding to 2 singular points (say  $s_1$  and  $s_2$ ). By the construction of the Board we see that the remaining two regions adjacent to  $v$  are  $E$ -regions (recall Figure 1); since crossing a center doesn't change an  $I$ -region to an  $E$ -region we see that both  $s_1$  and  $s_2$  are saddles.

Moving out of  $R_{i-1}$  by crossing  $s_1$ , we arrive at a region labelled  $E$ ; thus crossing the saddle has the effect of changing a single inessential curve into two parallel essential curves bounding an annulus. Since the surface was essentially green prior to crossing  $s_1$ , the annulus between the parallel curves is essentially yellow<sup>5</sup>. Crossing  $s_2$  into  $R_i$  the label becomes  $I$ ; hence the 2 parallel curves are pinched together to become a single inessential curve. If the pinching is done inside the essentially yellow annulus (thus turning it into a disk) the surface becomes essentially green;

<sup>3</sup>It is possible for  $R_1$  to meet the left edge in a single vertex and likewise it is possible that  $R_n$  has only one point at which  $s = 1$ .

<sup>4</sup>Note that  $v$  was obtained from pinching an edge of the Board.

<sup>5</sup>That is, the annulus is yellow except perhaps for regions contained in a disk.

hence the boundary of the annulus is pinched outside the essentially yellow annulus. We obtain an essentially yellow once-punctured torus  $T$ . Since  $R_i$  is an  $I$ -region the boundary of  $T$  is inessential; hence  $g = 1$ .

But  $g = 1$  is the base case of the induction and was treated in the introduction, where we assumed that  $g > 1$ . This contradiction establishes Proposition 2.2.  $\square$

When playing Hex the border forms a 1-manifold; as a consequence the game cannot end in a tie. We prove this in our case:

**Proposition 2.4.** *Esmeralda wins.*

*Proof.* (We work on the Board.) First observe that there is exactly one region adjacent to each corner of  $[-1, 1] \times [-1, 1]$ . These regions corresponds to disjoint surfaces  $S_s^2$  and  $\Sigma_t$ ; hence these regions are all  $I$ -regions. By taking any point  $(s, t)$  with  $t \neq \pm 1$  and moving  $s$  towards  $-1$  or  $1$ , we see that the label becomes  $I$ ; in conclusion every region adjacent to the left ( $s = -1$ ) or to the right ( $s = 1$ ) is labelled  $I$ .<sup>6</sup>

However, by Proposition 2.2 there is no chain of  $I$ -regions connecting the left to the right. So the Border must separate the left from the right; by Proposition 2.1 the Border forms an embedded 1-manifold in  $(-1, 1) \times (-1, 1)$  and by the above it does not meet the left or the right, we see that the Border has 4 type of components<sup>7</sup>:

- (1) Simple closed curves.
- (2) Arcs connecting the bottom edge to itself.
- (3) Arcs connecting the top edge to itself.
- (4) Arcs connecting the bottom edge to the top edge.

In order to separate the left from the right a component of type (4) must exist; on one side of this component we see a chain of regions labelled  $E$ ; this shows that Esmeralda wins.  $\square$

Esmeralda's victory is given by a path of regions in the Board, say  $R_1, \dots, R_n$ ,  $n \geq 1$ , all labelled  $E$  and connecting the bottom of the board to its top.<sup>8</sup> We consider the corresponding regions in the Graphic, still denoted  $R_1, \dots, R_n$ . Observe that  $R_{i-1}$  still shares an edge with  $R_i$  ( $i = 2, \dots, n$ ).

### 3. THE WEAK REDUCTION

We complete the proof by finding a weak reduction; this is very much a standard argument is Cerf Theory, originally due to [3]. First we show:

**Lemma 3.1** (regarding  $E$ -regions). *Let  $S_s^2$  and  $\Sigma_t$  be surfaces corresponding to a region labelled  $E$ . Then some curve of  $S_s^2 \cap \Sigma_t$  bounds a meridian disk in  $U$  or  $V$ .*

*Proof.* Let  $\mathcal{C} \subset S_s^2$  denote the collection of essential<sup>9</sup> curves of  $S_s^2 \cap \Sigma_t$  as an embedded 1-manifold in  $S_s^2$ . By assumption,  $\mathcal{C} \neq \emptyset$ . Let  $c \subset \mathcal{C}$  be an outermost curve and  $D \subset S_s^2$  the outermost disk it bounds. Then in its interior  $D$  intersects  $\Sigma_t$  in a (possibly empty) collection of curves that are inessential in  $\Sigma_t$ ; a standard disk swap argument gets a disk disjoint from  $\Sigma_t$  with boundary  $c$ .  $\square$

<sup>6</sup>With not much work we can get much more: there are exactly 3 regions adjacent to the left edge, the two corner regions and a third region that meets the left in a single point that corresponds to the unique value of  $t$  for which  $(0, 0, 0, -1) \in \Sigma_t$ . It is easy to see that surfaces that correspond to the third region have a single inessential curve of intersection and are therefore labelled  $I$ ; similarly to the right.

<sup>7</sup>We note that distinct components may share points on the boundary of  $[-1, 1] \times [-1, 1]$ , however in  $(-1, 1) \times (-1, 1)$  they are disjoint.

<sup>8</sup> $R_1$  may have a single point on the bottom edge and  $R_n$  may have a single point on the top; Cf. Footnote 3.

<sup>9</sup>Essential on  $\Sigma_t$ .

Since  $R_1$  contains points with  $t$  arbitrarily close to  $-1$  (where  $\Sigma$  collapses to a spine of  $U$ ) it is easy to see that curves of  $S_s^2 \cap \Sigma_t$  bound meridians of  $U$ ; likewise curves of  $S_s^2 \cap \Sigma_t$  in  $R_n$  bound meridians of  $V$ . By Lemma 3.1 every region  $R_i$  has curve of  $S_s^2 \cap \Sigma_t$  that bound meridians of  $U$  or  $V$ . Let  $i$  be the lowest index so that  $R_i$  has a curve of  $S_s^2 \cap \Sigma_t$  bounds a meridian of  $V$ . We arrive at the following dichotomy:

- (1) ( $i = 1$ ) Surfaces corresponding to  $R_i$  contain curves of  $S_s^2 \cap \Sigma_t$  bounding a meridian in  $U$  and curves bounding a meridian in  $V$ .
- (2) ( $i > 1$ ) Surfaces corresponding to  $R_{i-1}$  contain curves of  $S_s^2 \cap \Sigma_t$  bounding a meridian in  $U$  and surfaces corresponding to  $R_i$  contain curves of  $S_s^2 \cap \Sigma_t$  bounding a meridian in  $V$ .

In Case (1) we directly see a weak reduction or reduction (if both meridian disks bound the same curve). (Recall that reduction implies weak reduction.)

In case (2), we note that crossing from  $R_{i-1}$  to  $R_i$  corresponds to crossing one critical point, either a saddle or a center. In either case the set of essential curves in  $S_s^2 \cap \Sigma_t$  corresponding to  $R_{i-1}$  can be isotoped to be disjoint from those corresponding to  $R_i$ ; hence  $\Sigma$  (reduces or) weakly reduces.

As we found a weak reduction, we completed the proof of Theorem 1.

*Remark 3.2.* If  $g = 1$  it is clearly impossible to find a weak reduction. Reading through the proof, we find exactly one place where the assumption  $g > 1$  was used: in the proof that Ivan can't win (Proposition 2.2). We conclude that if we run the Cerf-theoretic argument in that case, it is actually Ivan who will win and Esmeralda who will lose.

As a concluding remark we mention that it is quite possible that Waldhausen never intended to study Heegaard splittings of  $S^3$ , but rather prove the Poincaré Conjecture. If we replace  $S^3$  with a homotopy 3-sphere the argument above fails miserably, since the “weak reduction” we will obtain consists of immersed disks (small problem, in light of Papakiryakopulus’s work) that might intersect each other (and hence will not give a weak reduction at all, even if each disk is embedded). Even if these problems is miraculously overcome, the best we can hope for is a “reduction” of the Heegaard surface via an immersed sphere that intersects the Heegaard surface in a single (probably not simple) closed curve. This is equivalent to the following condition: *the intersection of the kernels of the two maps induced on the fundamental group of  $\Sigma$  by its inclusion into  $U$  and  $V$  is non-trivial*. This is apparently not the right way to go: it was proven by J.R. Stallings in his paper “How Not to Prove the Poincaré Conjecture” [5].

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