

Separating incompressible surfaces and stabilizations of Heegaard splittings

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Abstract

We describe probably the simplest 3-manifold which contains closed separating incompressible surfaces of arbitrarily large genus. Two applications of this observation are given. (1) For any closed, orientable 3-manifold M and any integer $m > 0$, a surgery on a link in M of at most $2m + 1$ components will provide a closed, orientable, irreducible 3-manifold containing m disjoint, non-parallel, separating, incompressible surfaces of arbitrarily high genus. (2) There exists a 3-manifold M containing separating incompressible surfaces S_n of genus $g(S_n)$ arbitrarily large, such that the amalgamation of minimal Heegaard splittings of two resulting 3-manifolds cutting along S_n can be stabilized $g(S_n) - 3$ times to a minimal Heegaard splitting of M .



1. Introduction

1.1. Incompressible surfaces

Let M be a compact orientable 3-manifold and F be a compact 2-sided surface properly embedded in M . F is said to be compressible if either F bounds a 3-ball, or there is an essential, simple closed curve in F which bounds a disk in M ; otherwise, F is said to be incompressible.

The study of incompressible surfaces is a central topic in 3-manifold theory. A natural question is that if M has incompressible surfaces $\{F_i\}$, how many disjoint

non-parallel F_i 's can there be? What can we say about their genus and their separability?

The most basic result for the question is the Haken–Kneser Finiteness Theorem which claims that for any given M , there exists an integer $c(M)$, such that any collection of pair-wise disjoint, non-parallel, closed, incompressible surfaces in M has at most $c(M)$ components.

In this paper, for any m , we will construct a closed orientable irreducible 3-manifold containing m non-parallel closed incompressible separating surfaces of arbitrarily high genus.

Let $L = k_1 \cup \dots \cup k_m$ be a link in a compact 3-manifold M with m components. We denote by M_L the manifold $M - \text{int}(N(k_1) \cup \dots \cup N(k_m))$, where $N(k_i)$ is a regular neighbourhood of k_i . Let T_i be the boundary of $N(k_i)$, and r_i be a slope on T_i , $i = 1, \dots, m$. We denote by $M_L(r_1, \dots, r_m)$ the manifold obtained by attaching m solid tori J_1, \dots, J_m to M_L along T_1, \dots, T_m so that r_i bounds a disk in J_i , $i = 1, \dots, m$.

THEOREM 1. *Let M be a closed, orientable 3-manifold. For any given integer m , there exist a link $L = k_0 \cup k_1 \cup \dots \cup k_{2m}$ in M and slopes r_i on T_i , $i = 0, 1, \dots, 2m$, such that $M_L(r_0, r_1, \dots, r_{2m})$ is irreducible and contains m non-parallel separating closed incompressible surfaces of arbitrarily high genus.*

Remarks. (1) In 1969, using the stair-construction, Jaco [2] constructed closed incompressible surfaces of arbitrarily high genus, in 3-manifold $F_g \times S^1$, where $g > 1$ and F_g is the closed oriented surface of genus g . Shortly afterwards, this stair construction has been generalized to any surface bundle $F_g \times_\phi S^1$, $g > 1$, with the first Betti number at least two [6]. All surfaces here are non-separating.

(2) In 1970, Lyon [4] constructed a knot complement which contains closed incompressible surfaces of arbitrarily high genus. His construction, which is obtained first by a connected sum, then a satellite operation and finally another connected sum, is complicated.

(3) In 1996, Qiu [7] showed that a handlebody of genus at least two contains separating incompressible surfaces S of arbitrarily high genus (as did Howard independently). This construction only gives surfaces with boundary which are boundary-compressible.

(4) Moreover, none of three constructions above gives irreducible 3-manifolds containing two disjoint non-parallel incompressible surfaces of arbitrarily high genus.

1.2. Heegaard Splittings

A connected 3-manifold H is called a compression body if it is obtained by attaching some 1-handles to $F \times [0, 1]$ along $F \times \{1\}$ or 3-balls B^3 , where F is a closed surface. We write $\partial_- H = F \times \{0\}$, $\partial_+ H = \partial H - \partial_- H$. If there is a closed surface S in a 3-manifold M which separates M into two compression bodies H_1 and H_2 such that $S = \partial_+ H_1 = \partial_+ H_2$, then we say that M has a Heegaard splitting $M = H_1 \cup_S H_2$.

A Heegaard splitting $M = H_1 \cup_S H_2$ is said to be stabilized if there are properly embedded disks $D_1 \subset H_1$ and $D_2 \subset H_2$ such that ∂D_1 intersects ∂D_2 in only one point in S ; otherwise, it is said to be unstabilized.

A Heegaard splitting $M = H_1 \cup_S H_2$ is said to be minimal if the genus of S is minimal among all Heegaard splittings of M . This genus of S is also called the

Heegaard genus of M . If a Heegaard splitting is minimal, then the Heegaard splitting is unstabilized:

Suppose that $M = M_1 \cup_F M_2$ and $M_i = H_1^i \cup H_2^i$ is a Heegaard splitting of M_i , $i = 1, 2$, such that $F = \partial_- H_2^1 = \partial_- H_1^2$. Then M has a Heegaard splitting as follows:

$$M = (\partial_- H_1^1 \times I) \cup \{1\text{-handles in } H_1^1 \text{ and } H_1^2\} \cup \{2\text{-handles in } H_2^1 \text{ and } H_2^2\}.$$

This Heegaard splitting is called an **amalgamation** of the two Heegaard splittings $M_1 = H_1^1 \cup H_2^1$ and $M_2 = H_1^2 \cup H_2^2$ along $F = \partial_- H_2^1 = \partial_- H_1^2$.

THEOREM 2. *There exists a 3-manifold M containing separating incompressible surfaces S_n of arbitrarily large genus $g(S_n)$, such that the amalgamation of minimal Heegaard splittings of two resulting 3-manifolds along S_n can be stabilized $g(S_n) - 3$ times to a minimal Heegaard splitting of M .*

Theorem 2 was motivated by a conjecture of Gordon which claims that an amalgamation of the two unstabilized Heegaard splittings along S^2 is unstabilized. [3, problem 3-91]. Theorem 2 suggests the following:

GENERALIZED GORDON'S CONJECTURE. *An amalgamation of two unstabilized Heegaard splittings along a closed incompressible surface of genus g can be stabilized at most g times.*

The proofs of both Theorem 1 and Theorem 2 rely on an observation which will be presented as Proposition 1 in Section 2. Examples provided by Proposition 1 are probably the simplest 3-manifolds containing separating incompressible surfaces of arbitrary high genus. Theorem 1 and Theorem 2 will be proved by in Sections 3 and 4 respectively. The proof of Theorem 1 invokes results from [1], [5] and [8]. The proofs of Proposition 1 and Theorem 2 are self-contained.

2. Incompressible surfaces in a complement of a link in $F \times [0, 1]$

PROPOSITION 1. *Let F be an orientable, closed surface of genus at least two and let c be a non-separating, simple closed curve on F . Then $(F \times [0, 1])_{k_1 \cup k_2}$ contains separating, closed, incompressible surfaces of arbitrarily high genus, where $k_1 = c \times t_1$ and $k_2 = c \times t_2$, $0 < t_1 < t_2 < 1$.*

Proof. Let $N(c)$ be a regular neighbourhood of c on F . Then $N(c)$ is an annulus. We denote by c_0 and c_1 the two boundary components of $N(c)$. Let $n \geq 2$ be an integer, and $x_0 = 0 < x_1 = 1/8 < \dots < x_n = 7/8 < x_{n+1} = 1$. Then in $F \times [0, 1]$, the surface $F \times \{x_i\}$ intersects the annulus $c_j \times [0, 1]$ in the simple closed curve $c_j \times \{x_i\}$, where $j = 0, 1, i = 0, 1, \dots, n + 1$. We denote by $c_{j,i}$ the simple closed curve $c_j \times \{x_i\}$.

It is easy to see that there are $n - 1$ pair-wise disjoint annuli A_1^*, \dots, A_{n-1}^* properly embedded in $N(c) \times [0, 1]$ such that $\partial A_i^* = c_{0,i} \cup c_{1,i+1}$, $i = 1, 2, \dots, n - 1$. We denote by F_i the surface $F \times \{x_i\} - \text{int}(N(c) \times [0, 1])$. Now let $R_n = \bigcup_{i=1}^n F_i \cup \bigcup_{l=1}^{n-1} A_l^*$. Then $\partial R_n = c_{0,n} \cup c_{1,1}$. (As in Fig. 1.)

Next we choose $b > 0$ small enough so that there is an embedding $R_n \times [-b, b] \subset F \times [0, 1]$ such that for each $x \in R_n$, $x \times \{0\}$ is mapped to x and $x \times [-b, b]$ is embedded isometrically into the fiber $[0, 1]$ containing x . By construction, $\partial(R_n \times [-b, b])$, denoted by S_n , is separating in $F \times I$, and $g(S_n) = 2n(g(F) - 1) + 1$. (As in Fig. 1.)

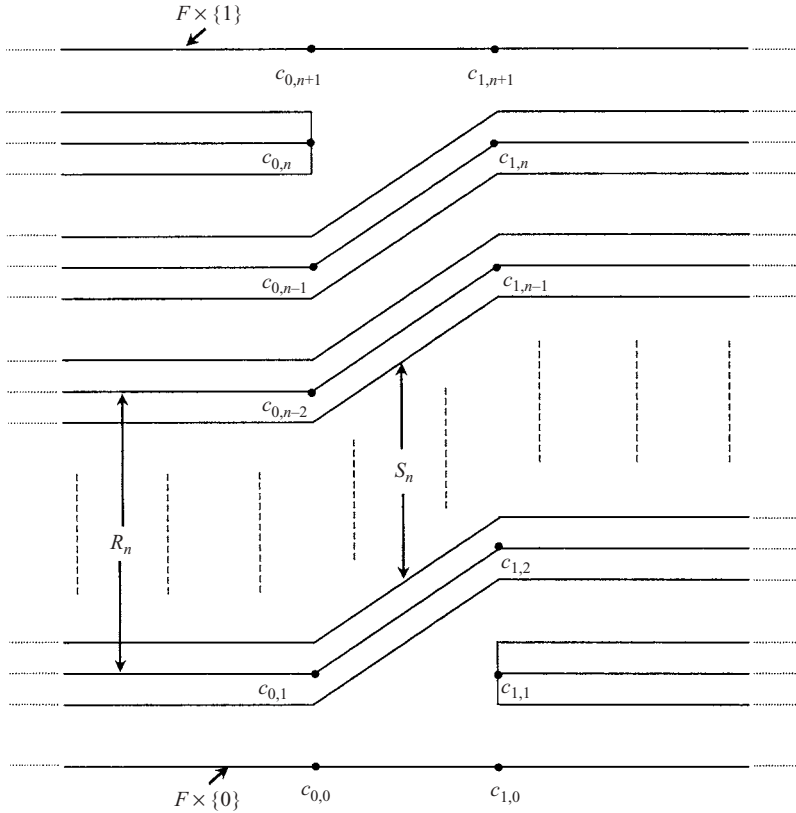


Fig. 1.

For each $c_{j,i} \subset R_n$, denote $c_{j,i} \times \{-b\}$ and $c_{j,i} \times b$ by $c_{j,i,-}$ and $c_{j,i,+}$ respectively. For $j = 0, 1$, let $A_{j,i}$ be the annulus in $c_j \times [0, 1]$ bounded by $c_{j,i,+}$ and $c_{j,i+1,-}$, if $i = 1, \dots, n - 1$; and by $c_{j,0}$ and $c_{j,1,-}$ if $i = 0$; and by $c_{j,n,+}$ and $c_{j,n+1}$ if $i = n$ (as in Fig. 2).

Now let k_1^n be the knot in $R_n \times [-b, b]$ obtained by pushing $c_{0,n}$ slightly into $\text{int}(R_n \times [-b, b])$, and k_2^n be the knot obtained by pushing $c_{1,1}$ slightly into $\text{int}(R_n \times [-b, b])$. Let $L_n = k_1^n \cup k_2^n$. Since $x_1 = 1/8, x_n = 7/8$ for any integer $n, k_1^{n_1} = k_1^{n_2}$ and $k_2^{n_1} = k_2^{n_2}$ even if $n_1 \neq n_2$. Thus we denote by k_1 the knot k_1^n, k_2 the knot k_2^n , and L the link L_n . (Note that the link $L = k_1 \cup k_2$ described here and the link $k_1 \cup k_2$ described in Proposition 1 are isotopic in $F \times [0, 1]$.)

Now $T_i = \partial N(k_i), i = 1, 2$, are as indicated in Fig. 2.

We shall prove that S_n is incompressible in $(F \times [0, 1])_L$ by proving the following two claims, where the proof of Claim 1 is an essential part of the proof of Claim 2, and is also used in the proof of Proposition 3.

Claim 1. S_n is incompressible in $(R_n \times [-b, b])_L$.

Proof. By construction, for any integer $n \geq 2, c_{0,n}$, together with the longitude slope l_1 on $T_1 = \partial N(k_1)$, bound an annulus B_1 , and $c_{1,1}$, together with the longitude slope l_2 on $T_2 = \partial N(k_2)$, bound an annulus B_2 (Fig. 2).

Now suppose that S_n is compressible in $(R_n \times [-b, b])_L$. Let D be a compressing disk of S_n such that $|D \cap (B_1 \cup B_2)|$, the number of components of $D \cap (B_1 \cup B_2)$,

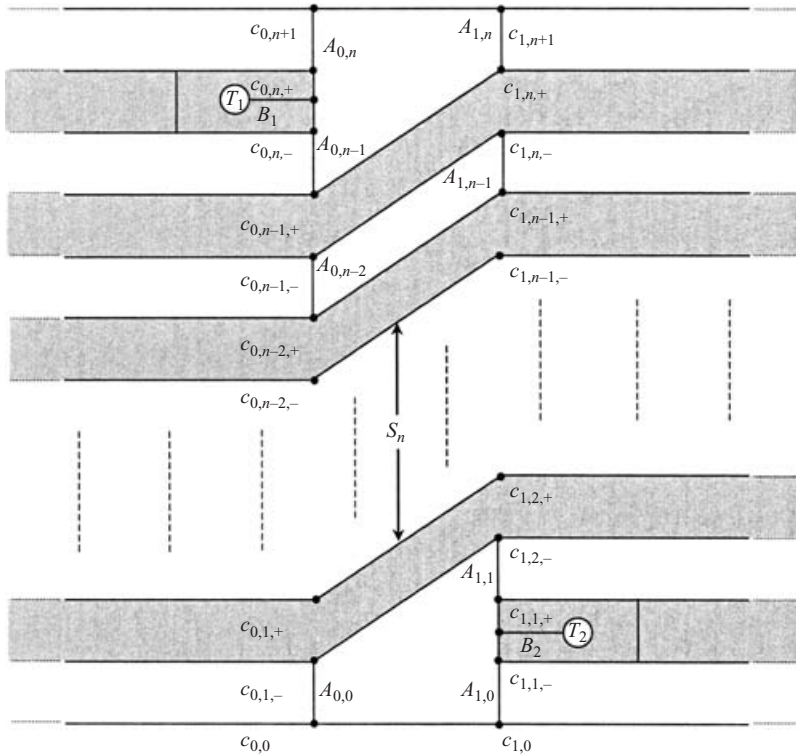


Fig. 2.

is minimal among all such disks. By definition, ∂D is essential in S_n . Note that $|D \cap (B_1 \cup B_2)| \neq \emptyset$. Otherwise, ∂D can be isotoped into either $R_n \times \{-b\}$ or $R_n \times \{b\}$, and one of $R_n \times \{-b\}$ and $R_n \times \{b\}$ is compressible in $(R_n \times [-b, b])_L$, which is not possible.

Suppose one component of $D \cap (B_1 \cup B_2)$ is a simple closed curve. Since both the centrelines of B_1 and B_2 are essential, this simple closed curve must bound a disc in $B_1 \cup B_2$. Then a standard argument shows that there is a compressing disk D_0 of S_n such that $|D_0 \cap (B_1 \cup B_2)| < |D \cap (B_1 \cup B_2)|$. Thus we may assume that each component of $D \cap (B_1 \cup B_2)$ is an arc, the two end points of which lie in one of $c_{0,n}$ and $c_{1,1}$. Without loss of generality, we assume that $D \cap B_1 \neq \emptyset$. Let a_1 be an arc in $D \cap B_1$ which, together with an arc a_2 on $c_{0,n}$, bounds a disk D^* in B_1 such that $\text{int} D^*$ is disjoint from D . We denote by a_3 and a_4 the two components of $\partial D - \partial a_2$. Then $d_1 (= a_1 \cup a_3)$ and $d_2 (= a_2 \cup a_4)$ bound disks D_1 and D_2 respectively in $(R_n \times [-b, b])_L$. Since ∂D is essential in S , one of d_1 and d_2 , say d_1 , is essential. But $|D_1 \cap (B_1 \cup B_2)| < |D \cap (B_1 \cup B_2)|$, a contradiction.

We denote by E the manifold $F \times [0, 1] - \text{int}(R_n \times [-b, b])$ below.

Claim 2. S_n is incompressible in E .

Proof. Suppose that S_n is compressible in E . Let D be a compressing disk of S_n in E such that $|D \cap (\bigcup_{j=0}^1 \bigcup_{i=0}^n A_{j,i})|$ is minimal among all such disks. Note that $|D \cap (\bigcup_{j=0}^1 \bigcup_{i=0}^n A_{j,i})| \neq \emptyset$. Otherwise, for some i , $F \times \{x_i\}$ is compressible in $F \times [0, 1]$. By the minimizing assumption, $D \cap (\bigcup_{j=0}^1 \bigcup_{i=0}^n A_{j,i})$ contains no circle component (as

indicated in the proof of Claim 1). Suppose a is an arc component of $\partial D \cap A_{j,i}$, $i=0, n$, then the two end points of a must lie in one component of $\partial A_{j,i}$, since $\partial D \subset S_n$ and $\partial A_{j,i}$ has only one component in S_n . However by the proof of Claim 1, we can reduce the number of components of $D \cap (\bigcup_{j=0}^1 \bigcup_{i=0}^n A_{j,i})$. That means that $D \cap (\bigcup_{j=0}^1 A_{j,0} \cup A_{j,n}) = \emptyset$, and for each arc $a \in D \cap A_{j,i}$, the two end points of a lie in distinct components of $\partial A_{j,i}$. Let a_1 be a component of $D \cap (\bigcup_{j=0}^1 \bigcup_{i=1}^{n-1} A_{j,i})$ which, together with an arc a_2 on ∂D , bounds a disk D^* in D such that $\text{int}D^*$ is disjoint from $\bigcup_{j=0}^1 \bigcup_{i=1}^{n-1} A_{j,i}$. Without loss of generality, we assume that $a_1 \subset A_{j,l}$. Then one of the two end points of a_1 lies in $c_{j,l,+}$, and the other lies in $c_{j,l+1,-}$. Since $c_{0,l,+} \subset R_n \times \{b\}$, $c_{0,l+1,-} \subset R_n \times \{-b\}$, and any arc with one end in $F_n \times \{b\}$ and the other in $R_n \times \{-b\}$ must cross either $c_{1,1,-}$ or $c_{0,n,+}$, we have $a_2 \cap (c_{1,1,-} \cup c_{0,n,+}) \neq \emptyset$. But we have proved that $D \cap (\bigcup_{j=0}^1 A_{j,0} \cup A_{j,n}) = \emptyset$, a contradiction.

By Claim 1 and Claim 2, S_n is incompressible in $(F \times [0, 1])_L$. We have finished the proof of Proposition 1.

The irreducibility of $(F \times [0, 1])_L$ follows directly from the lemma below whose proof is direct.

LEMMA 1. *Suppose M is an irreducible 3-manifold and L is a link in M . If each component of L is essential in M , then M_L is irreducible.*

3. The proof of Theorem 1

Let M be a closed, orientable 3-manifold. First we prove the following:

PROPOSITION 3. *There is a knot k in M such that:*

- (1) M_k is irreducible and boundary irreducible;
- (2) M_k contains a separating closed, incompressible surface of genus at least two.

Proof. Let $H_1 \cup_S H_2$ be a Heegaard splitting of M with $g(S)$ an odd integer. Since ∂H_1 contains no sphere components, by [5], there is a properly embedded arc a in H_1 such that $H_1 - \text{int}N(a)$ is irreducible and boundary irreducible. Then $H = H_2 \cup N(a)$ is a handlebody of genus an even integer at least 2. Note that H is homeomorphic to the product $S \times [0, 1]$, where S is a compact orientable surface with ∂S connected. Let $c^* = \partial S \times \{0\}$. Then c^* is an essential separating simple closed curve on ∂H and $\partial H - c^*$ is incompressible in H .

Let k be the knot in H obtained by pushing c^* slightly into $\text{int}H$. Since k is essential in H , $H_k = H - \text{int}N(k)$ is irreducible by Lemma 2.

Now we verify that ∂H is incompressible in $H - \text{int}N(k)$. Suppose that ∂H is compressible in H_k . Let D be a compressing disk of ∂H such that $|\partial D \cap c^*|$ is minimal among all such disks. Since $\partial H - c^*$ is incompressible, $|\partial D \cap c^*| \neq \emptyset$. Since c^* , together with the longitude slope on $\partial N(k)$, bounds an annulus, by the proof of Claim 1, there is a compressing disk D' of ∂H , such that $|\partial D' \cap c^*| < |\partial D \cap c^*|$, a contradiction.

Since $\partial N(k)$ is also incompressible in H_k , H_k is boundary irreducible.

Now for the complement of k in M , $M_k = (H_1 - \text{int}N(a)) \cup_{\partial H} H_k$, we have ∂H is incompressible in both $H_1 - \text{int}N(a)$ and H_k , hence ∂H is incompressible in M_k . Since both $H_1 - \text{int}N(a)$ and H_k are irreducible and boundary irreducible, M_k is irreducible and boundary irreducible.

Clearly $g(\partial H) \geq 2$, we have proved Proposition 3.

We need two known results about Dehn surgery on knots.

Suppose M is an irreducible 3-manifold with a torus T in ∂M . Let r_1, r_2 be two slopes on T . We use $\Delta(r_1, r_2)$ to denote the minimal geometric intersection number of the two slopes.

PROPOSITION 4 ([1, theorem 2.4.3]). *Suppose that M is as above and that S is an incompressible surface in M . Suppose also that M contains an annulus with one boundary component in S and the other having slope l in T and that M is not homeomorphic to $T \times I$. Then S is incompressible in $M(r)$ if $\Delta(l, r) > 1$.*

PROPOSITION 5 ([8]). *Suppose that M is as above and that M is also boundary-irreducible. If $M(r_1)$ is reducible and $M(r_2)$ is boundary reducible, then $\Delta(r_1, r_2) = 0$.*

Proof of Theorem 1. Below we denote the knot k from Proposition 3 by k_0 . Now M_{k_0} contains a closed, incompressible surface $F = \partial H$ of genus at least two. By the construction of k_0 , the simple closed curve c^* on F , together with the longitude slope l_0 on $T_0 = \partial N(k_0)$, bounds an annulus in M_{k_0} . Let r_0 be a slope on T_0 such that $\Delta(r_0, l_0) > 1$. Since H_{k_0} is boundary irreducible and is not homeomorphic to the product. By Proposition 4, F is incompressible in $M_{k_0}(r_0)$.

We are going to show that $M_{k_0}(r_0)$ is irreducible. Since F is compressing in $H_{k_0}(l_0)$ and $\Delta(r_0, l_0) \geq 2$, $H_{k_0}(r_0)$ is irreducible by Proposition 5. Since both $H_{k_0}(r_0)$ and $H_1 - \text{int}N(a)$ are boundary incompressible and irreducible, $M_{k_0}(r_0) = (H_1 - \text{int}N(a)) \cup_F H_{k_0}(r_0)$ is irreducible.

To fit the symbols and notions used in the Proof of Proposition 1, we first assume that $m = 1$.

Denote $M_{k_0}(r_0)$ by M_* and suppose that we have the incompressible surface F of genus at least 2 in M_* . Let $F \times [0, 1]$ be a regular neighbourhood of F in M_* . By Proposition 1, there is a link $L = k_1 \cup k_2$ of two components in $F \times [0, 1]$ such that S_n constructed in Proposition 1 is incompressible in $(F \times [0, 1])_L$.

Since $\partial(F \times [0, 1])$ is incompressible in M_* , it follows that S_n is incompressible in $(M_*)_L$. Since each component of L is essential in M_* , $(M_*)_L$ is irreducible by Lemma 2.

Note that by the proof of Claim 1 in Proposition 1, $c_{0,n} \subset S_n$, together with the longitude slope l_1 on T_1 , bounds an annulus B_1 , and $c_{1,1} \subset S_n$, together with the longitude slope l_2 on T_2 , bounds an annulus B_2 .

By applying both Propositions 4 and 5 twice as before, we can choose the slope r_i on T_i , $i = 1, 2$, so that S_n is incompressible in the manifold $(M_*)_L(r_1, r_2) = M_{k_0 \cup k_1 \cup k_2}(r_0, r_1, r_2)$, which is irreducible. We have proved the case $m = 1$.

Now we discuss the case for any integer $m > 0$.

Once the incompressible surface F is available, let F^1, \dots, F^m be m disjoint parallel copies of F , and $N(F^1), \dots, N(F^m)$ be their disjoint regular neighbourhoods. Now construct an incompressible surface $S_{n_i}^i$ in F^i as in Proposition 1, and then do Dehn fillings $2m$ times to get a closed manifold. By the same argument as in the Proof above, the filling can be chosen so that each $S_{n_i}^i$ is incompressible in the resulting manifold and the resulting manifold is irreducible. If we choose all n_i different, then the genera of all the $S_{n_i}^i$ are different; hence all $S_{n_i}^i$, $i = 1, \dots, m$, are non-parallel.

This completes the proof of Theorem 1.

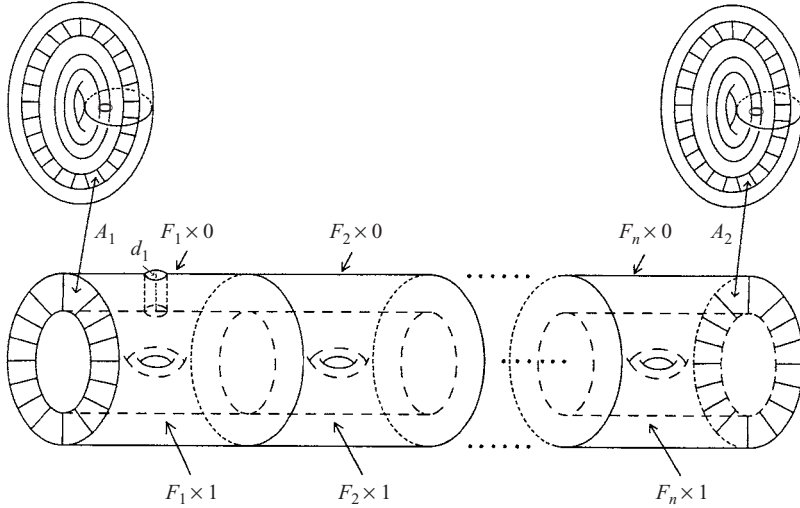


Fig. 3.

As a corollary of the proof of Theorem 1, if M contains an incompressible surface of genus at least 2, then we need only do surgery on a link of two components to get a 3-manifold containing separating incompressible surfaces of arbitrarily high genus.

4. The proof of Theorem 2

Let F in Proposition 1 be a closed surface of genus exactly two. Then $S_n \subset M = F \times [0, 1] - \text{int}N(L)$ in Proposition 1 is the closed separating incompressible surface of genus $2n + 1$. We denote by M_1, M_2 the two components of $M - S_n$. Without loss of generality, we assume that $T_1, T_2 \subset M_1$ (then M_1 is the shaded part in Fig. 2). Note that A_1 and A_2 (as in Fig. 2) separate M_1 into three parts $T_1 \times [0, 1], T_2 \times [0, 1]$ and $G_1^* \times [0, 1]$, where G_1^* is the surface of genus n with two boundary components, which can be obtained by gluing n copies of 2-punctured tori in an obvious way. We labeled those n copies of 2-punctured tori in $G_1^* \times k$ as $F_1 \times k, \dots, F_n \times k, k = 0, 1$ as in Fig. 3. Similarly the two annuli $A_{1,1}$ and $A_{0,n-1}$ (as in Fig. 2) separate M_2 into three parts: two copies of $F \times [0, 1]$, and $G_2^* \times [0, 1]$, where G_2^* is the surface of genus $n - 1$ with two boundary components, which can be obtained by gluing $n - 1$ copies of 2-punctured tori. To indicate precisely the identification of M_1 and M_2 , those $n - 1$ copies of 2-punctured tori was labeled as $F_1 \times 1, \dots, F_{n-1} \times 1$ in $G_2^* \times 0$, and as $F_2 \times 0, \dots, F_n \times 0$ in $G_2^* \times 1$, see Fig. 4. Let d_1 be a properly embedding arc in $G_1^* \times [0, 1]$ as in Fig. 3 and d_2 be an embedding arc in $G_2^* \times [0, 1]$ as in Fig. 4. Then we identify $N(d_i)$ with $D_i \times [0, 1]$, where D_i is a sub-disc of G_i . We also denote $G_i = G_i^* - \text{int}D_i, i = 1, 2$.

LEMMA 6. $M_i - \text{int}N(d_i)$ is a compression body, $i = 1, 2$.

Proof. Let l_1 and l_2 be two arcs on $G_1 \times [0, 1]$ connecting ∂A_1 to $\partial N(d_1)$, and l_3 and l_4 be two arcs on $G_1 \times \{0\}$ connecting ∂A_2 to $\partial N(d_1)$ as in Fig. 5.

Let $D_{1\#}$ (resp. D_{1*}) be a separating disk in $M_1 - \text{int}N(d_1)$, which consists of three parts: $l_1 \times [0, 1], l_2 \times [0, 1]$, and the sub-disc of A_1 which is shaded (resp. $l_3 \times [0, 1], l_4 \times [0, 1]$, and the sub-disc of A_2). Those two discs are indeed located in

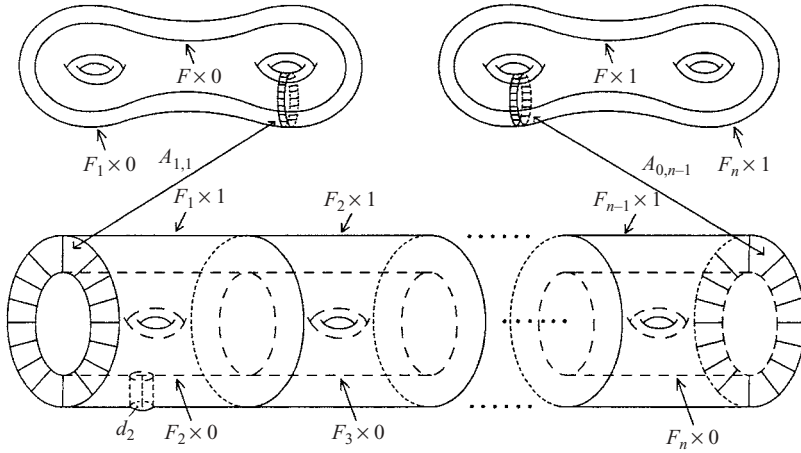


Fig. 4.

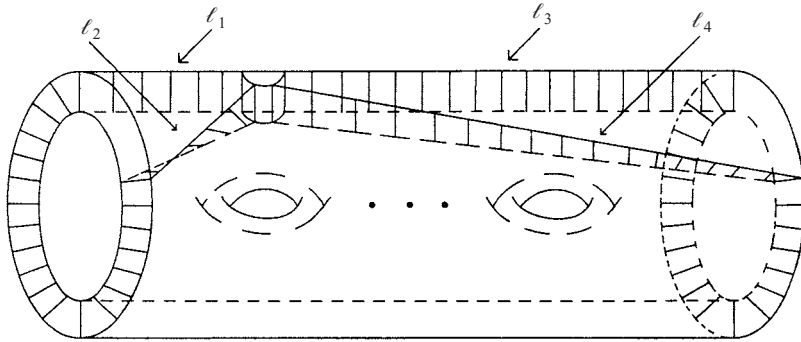


Fig. 5.

$G_1 \times [0, 1] - \text{int}N(d_1)$, indicated as in Fig. 5. Let $D_{2\#}$ (resp. D_{2*}) be a separating disk in $M_2 - \text{int}N(d_2)$ constructed in the similar way.

Let a_1, \dots, a_{2n} are $2n$ proper arcs in G_1 as in Fig. 6(a) and b_1, \dots, b_{2n-2} are $2n - 2$ proper arcs in G_2 as in Fig. 6(b). Without loss of generality, we assume that these l_i 's and a_j 's are disjoint, and so on.

We denote by $D_{1,j}$ the disk $a_j \times [0, 1]$, $D_{2,j}$ the disk $b_j \times [0, 1]$.

Then $D_{1\#} \cup D_{1*} \cup_{i=1}^{2n} D_{1,i}$ separates M_1 into $T_1 \times [0, 1]$, $T_2 \times [0, 1]$ and a 3-ball. Thus $M_1 - \text{int}N(d_1)$ is a compression body. Similarly $D_{2\#} \cup D_{2*} \cup_{i=1}^{2n-2} D_{2,i}$ separates M_2 into two copies of $F \times [0, 1]$ and a 3-ball and $M_2 - \text{int}N(d_2)$ is a compression body.

Note

$$(M_i - \text{int}N(d_i)) \cup N(d_i) = (M_i - \text{int}N(d_i)) \cup ((S_n \times I) \cup N(d_i)),$$

the right-hand side provides a Heegaard splitting of M_i . Since M_i is boundary irreducible, $(M_i - \text{int}N(d_i)) \cup N(d_i)$ is minimal. Now $((M_2 - \text{int}N(d_2)) \cup N(d_1)) \cup ((M_1 - \text{int}N(d_1)) \cup N(d_2))$ is the amalgamation of the two Heegaard splittings of genus $2n + 3$ along S_n . It is easy to see that the basis disks of $(M_2 - \text{int}N(d_2)) \cup N(d_1)$ are $D_{2\#}, D_{2*}, D_1 \times t_1, D_{2,1}, \dots, D_{2,2n-2}$, and the basis disks of $(M_1 - \text{int}N(d_1)) \cup N(d_2)$ are $D_{1\#}, D_{1*}, D_2 \times t_2, D_{1,1}, \dots, D_{1,2n}$.

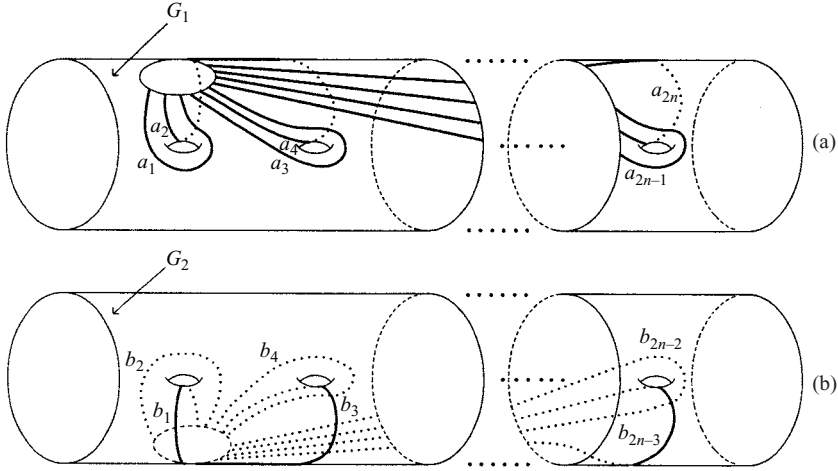


Fig. 6.

LEMMA 7.

- (1) $|D_{1,i} \cap D_{2,i}| = 1$ for $i = 1, \dots, 2n - 2$.
- (2) $|D_{1,i} \cap D_{2,i-2}| = 1$ for $i = 3, \dots, 2n$.
- (3) $D_{1,i} \cap D_{2,j} = \emptyset$ for each $1 \leq i \leq 2n, 1 \leq j \leq 2n - 2$ with $j \neq i - 2, i$.

Proof. First identify G_2 in Fig. 6(b) with $\bigcup_{i=1}^{2n-1} F_i \subset G_1$ in Fig. 6(a) in an obvious way, denote the images of the b_j on G_1 still by b_j for $j = 1, \dots, 2n - 2$. Then identify G_2 in Fig. 6(b) with $\bigcup_{i=2}^n F_i \subset G_1$ in Fig. 6(a) in an obvious way, and denote the images of the b_j on G_1 by b'_j for $j = 1, \dots, 2n - 2$. It is easy to see that

$$|a_i \cap b_i| = 1, \quad a_i \cap b_j = \emptyset, \quad i = 1, \dots, 2n - 2, \quad i \neq j. \quad (*)$$

$$|a_i \cap b'_{i-2}| = 1, \quad a_i \cap b_j = \emptyset, \quad i = 3, \dots, 2n, \quad j \neq i - 2. \quad (**)$$

By definition, $D_{1,i} = a_i \times [0, 1]$ and $D_{2,j} = b_j \times [0, 1]$. Then $\partial D_{1,i}$ is a union of two arcs on $\partial D_1 \times [0, 1]$, and the arc $a_i \times 0$ on $\bigcup_{k=1}^i F_k \times 0$ and the arc $a_i \times 1$ on $\bigcup_{k=1}^i F_k \times 1$; $\partial D_{2,j}$ is a union of two arcs on $\partial D_2 \times [0, 1]$, and the arc $b_j \times 0$ on $\bigcup_{k=1}^j F_k \times 1$ and the arc $b_j \times 1$ on $\bigcup_{k=2}^{j+1} F_k \times 0$. Then Lemma 7 follows from the following three facts, and formulas (*) and (**):

- (1) the two arcs on $\partial D_1 \times [0, 1]$ and the two arcs on $\partial D_2 \times [0, 1]$ are disjoint;
- (2) $a_i \times 0$ and $b_j \times 1$ on $\bigcup_{k=1}^n F_k \times 0$ meet exactly as a_i and b_j meet in G_1 ;
- (3) $a_i \times 1$ and $b_j \times 0$ on $\bigcup_{k=1}^n F_k \times 1$ meet exactly as a_i and b'_j meet in G_1 .

Let M be a compact 3-manifold with boundary, D, D' be two proper discs in M and $\alpha \subset \partial M$ be an arc which meets ∂D and $\partial D'$ exactly in its two ends. We use $D \#_\alpha D'$ to denote the proper disc obtained from the band connected sum of D and D' along α . We list the following simple facts as:

LEMMA 8.

- (1) D can be properly isotoped in its small regular neighbourhood so that $D \cap (D \#_\alpha D') = \emptyset$.
- (2) Let $C \subset \partial M$ be circle which meets both ∂D and $\partial D'$ transversely exactly in one point. Let $\alpha \subset C$ be an arc connecting ∂D and $\partial D'$. Then $(D \#_\alpha D') \cap C = \emptyset$.

Proof of Theorem 2. Let

$$D_{1,2i-1}^* = D_{1,1} \#_{\alpha_1} D_{1,3} \#_{\alpha_3} \cdots \#_{\alpha_{2j-3}} D_{1,2j-1} \#_{\alpha_{2j-1}} \cdots \#_{\alpha_{2i-3}} D_{1,2i-1}$$

and

$$D_{1,2i}^* = D_{1,2} \#_{\alpha_2} D_{1,4} \#_{\alpha_4} \cdots \#_{\alpha_{2j-2}} D_{1,2j} \#_{\alpha_{2j}} \cdots \#_{\alpha_{2i-2}} D_{1,2i},$$

where α_j is an arc in $\partial D_{2,j}$ connecting $\partial D_{1,j}$ to $\partial D_{1,j+2}$. Such an arc exists by Lemma 7 (1) and (2). Then

$$D_{1,2i-1}^* = D_{1,2i-3}^* \#_{\alpha_{2i-3}} D_{1,2i-1}, \quad D_{1,2i}^* = D_{1,2i-2}^* \#_{\alpha_{2i-2}} D_{1,2i}.$$

Applying Lemma 8 (1) finitely many times, we can assume that all $D_{1,i}^*$'s are disjoint. By our construction, Lemma 7 and Lemma 8 (2) we have

$$|D_{1,i}^* \cap D_{2,i}| = 1, \quad i = 1, \dots, 2n - 2 \quad D_{1,i} \cap D_{2,j} = \emptyset \quad \text{if } i \neq j.$$

That means that the amalgamation can be stabilized $2n - 2 = g - 3$ times to a new Heegaard splitting of M of genus 5. It is easy to see that this Heegaard splitting is a minimal Heegaard splitting of M .

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