A SHORT PROOF OF BING'S CHARACTERIZATION OF S^3.

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Let M be a closed orientable 3-manifold. We assume familiarity with the basic notions of irreducible and prime 3-manifolds and the basic results about Heegaard splittings of compact 3-manifolds (see, *e.g,* [5]). In [1, Theorem 1] Bing (working on the Poincaré Conjecture) proved (in his words):

Theorem 1 (Bing)**.** *A compact, connected 3-manifold M is topologically S^3 if each simple closed curve in M lies in a topological cube in M.*

A knot $k \subset M$ (*i.e.* a smooth embedding of the circle into M) is called *irreducible* if its exterior $E(k) = M \setminus N(k)$ is an irreducible 3-manifold. We will prove that every 3-manifold admits an irreducible knot. If M is not S^3 it is easy to see that a knot contained in a ball (say B) is reducible (by considering the boundary of B); thus Theorem 1 follows from the existence of irreducible knots. In [3, Theorem 8.1] Jaco and Rubinstein gave a very short proof of existence of irreducible knots in irreducible manifolds, but their proof relies on the existence of 0-efficient triangulations. The purpose of this note is giving an easy proof of this fact.

Proof. **Step One:** M is prime or S^3. If $g(M)$ (the Heegaard genus of M) is at most one then take k to be a knot on a Heegaard torus with $E(k)$ a Seifert fibered space over the disk with 2 exceptional fibers. If $g(M) \geq 2$ then M is irreducible; let $M = V_1 \cup_\Sigma V_2$ be a minimal genus Heegaard splitting of M. By Waldhausen [6] (see also [5, Theorem 3.8]) Σ is irreducible. Take k to be a core of a 1-handle in V_1. Then Σ is an irreducible Heegaard surface for $E(k)$; Haken [2] (see also [5, Theorem 3.4]) showed that every Heegaard surface of a reducible manifold is reducible; hence $E(k)$ is irreducible.

Remark 2. In Case One, $\partial E(k)$ is incompressible. This is clear when $g(M) \leq 1$. When $g(M) \geq 2$ if $\partial E(k)$ compressed then irreducibility of $E(k)$ would imply that $E(k)$ is a solid torus; but then $g(M) \leq 1$.

Step Two: M is composite. By Kneser [4] M has a decomposition as $M \cong M_1 \# \cdots \# M_n$ with M_i prime ($i = 1, \dots, n$). Let $k_i \subset M_i$ be the knot obtained in Step One, let $k = \#_{i=1}^n k_i \subset M$ be their connected sum, and let $A_j \subset E(k)$ be annuli that decompose k into its summands ($j = 1, \dots, n-1$). Let S be a sphere in $E(k)$. By isotopy, minimize $S \cap (\cup_{j=1}^{n-1} A_j)$; if this intersection is empty then S is contained in a component of $E(k)$ cut open along $\cup_{j=1}^{n-1} A_j$. This component is homeomorphic to $E(k_i)$ (for some i) and thus S bounds a ball by Step One. Assume that S cannot be isotoped away from the A_j; since $\chi(S) = 2$, S cut open along $\cup_{j=1}^{n-1} A_j$ has disk components, showing that the meridian for some K_i bounds a disk, contradicting Remark 2. Thus every sphere in $E(k)$ bounds a disk and k is an irreducible knot, completing the proof. ☐

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