Research Statement

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Introduction

An important problem in Complex Analysis of several variables is to find functions holomorphic in a domain $\Omega \subseteq \mathbb{C}^n$ and satisfying certain properties. Holomorphic extensions of a function holomorphic in a subspace such as a complex hyperplane in Ω is one such example. One main approach used to do this involves solving the inhomogeneous Cauchy Riemann equations, $\bar{\partial}u = \alpha$. In general, for a $\bar{\partial}$-closed $(p, q+1)$ form α we want to solve the $\bar{\partial}$-equation $\bar{\partial}u = \alpha$. The question whether u inherits smoothness properties of α is important in many contexts.

My research is concerned with the compactness of the $\bar{\partial}$-Neumann operator. The $\bar{\partial}$-Neumann problem for a smooth bounded domain $\mathbb{C}^n$ is the problem of inverting the complex Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ with the solution satisfying certain boundary conditions. When Ω is a smooth bounded pseudoconvex domain the existence of the solution to the $\bar{\partial}$-Neumann problem in the L^2-sense was given by Hörmander. The $\bar{\partial}$-Neumann operator N_q is the inverse of the complex Laplacian defined in this way. For a $\bar{\partial}$-closed $(p, q+1)$ form α, a formula due to J. J. Kohn shows that $\bar{\partial}^* N_{q+1}\alpha$ gives the solution u (of minimal norm) to $\bar{\partial}u = \alpha$. Therefore, regularity of N_q, in the sense of preservation of L^2-Sobolev spaces for all positive indices, gives regularity of the solution to the $\bar{\partial}$-equation. Compactness of the operator N_q implies the regularity of N_q. Other consequences of compactness of the $\bar{\partial}$-Neumann operator include the Fredholm theory of Toeplitz operators. Thus it is important to know properties of the boundary that are necessary and sufficient for compactness of N_q.

Property$(\tilde{P}_q)$, which was introduced by McNeal, is a well known sufficient condition for compactness of the $\bar{\partial}$-Neumann operator N_q acting on q-forms. This generalizes Catlin's sufficient condition property(P_q). For a bounded pseudoconvex domain Ω we say that the boundary $b\Omega$ satisfies property(P_q) if there exists a family of uniformly bounded plurisubharmonic functions $\{\lambda\}$, defined in some neighborhood U_λ of the $b\Omega$, with arbitrarily large Hessian. Property$(\tilde{P}_q)$ replaces the uniform boundedness of these plurisubharmonic functions with uniformly bounded gradients, but in the metric induced by the Hessians of the functions. Equivalence of property$(\tilde{P}_q)$ and compactness of N_q is known for locally convexifiable domains in $\mathbb{C}^n$ and smooth bounded Hartogs domains in $\mathbb{C}^2$ ([2], [4]). For general smooth bounded pseudoconvex domains however it is not known, and in fact, may not hold. Probably some slightly weaker condition is needed to characterize compactness of the $\bar{\partial}$-Neumann operator. Given this situation, it is important to have compactness proofs that do not proceed through verifying property$(\tilde{P}_q)$. My work is in this direction.

Results

When the boundary of a domain Ω is strictly pseudoconvex the $\bar{\partial}$-Neumann operator is compact (in fact, much more is known). Other geometric conditions that imply compactness are finite type and in convex domains, absence of discs in the boundary. All these conditions also imply property(P_q). Straube ([10]) found other geometric conditions for domains in $\mathbb{C}^2$

that imply compactness. There however compactness is not proved by verifying property$(\tilde{P}_q)$. These geometric conditions are satisfied if the following holds. Denote by K the set of boundary points of infinite type; denote by $\mathcal{F}_Z^t$ the flow generated by a (real) vector field Z. There are constants $C_1, 1 \leq C_2 < 3/2$, such that for every $\epsilon > 0$ and for every $p \in K$, there is a (real) complex tangential vector field Z of unit length defined in some neighborhood of p, such that $\mathcal{F}_Z^t(B(p, C_1\epsilon^{C_2}))$ is relatively compact in the set of points of finite type. Using maximal estimates shown in Derridj ([3]) that hold in $\mathbb{C}^2$, Straube obtained a geometric proof independent of property$(\tilde{P}_q)$. This theorem without additional assumptions does not hold in higher dimensions. The main result we obtained is a generalization of this theorem to higher dimensions. The additional assumption needed in higher dimensions is that the vector fields Z at each point be in the eigenspace of the lowest eigenvalue of the Levi form. Compactness is proved by verifying a compactness estimate for $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$. As in the case of $\mathbb{C}^2$, to estimate the $\mathcal{L}^2$-norm of u near an infinite type point we express u in a patch which meets the boundary in a relatively compact subset of the finite type points plus the integral of the derivative of u in the direction of the vector field Z. The difference in the higher dimension arises in estimating the derivative of u in the direction of the vector field Z. In general, it is not true that complex tangential derivatives are controlled by $\overline{\partial}$ and $\overline{\partial}^*$ (so called maximal estimates fail in general in dimension $n > 2$). However, when the derivative is in a direction that lies in the eigenspace of the smallest eigenvalue of the Levi form (pointwise), then the methods of Derridj can be suitably adapted. One obtains estimates of the following flavor. There is a constant C such that for $\epsilon > 0$ and $t \in [0, \epsilon)$, we have

$$\int_{\mathcal{F}_Z^t(U\cap\Omega)} |Zu(y)|^2 \, dV(y) \quad \leq \quad C \left(\|\bar{\partial} u\|_0^2 + \|\bar{\partial}^* u\|_0^2 \right),$$

where U is a small neighborhood of a boundary point. There are overlap issues when one sums up these estimates, but these are handled in the same way as in $\mathbb{C}^2$, by a Vitali type covering argument.

As in the case for $\mathbb{C}^2$, a geometrically simple special case occurs when $b\Omega \setminus K$ satisfies a weak complex tangential cone condition, but here with the additional assumption that the eigenspace of the lowest eigenvalue is a smoothly varying subspace of the complex tangent space of $b\Omega$. This assumption is satisfied in particular when the Levi form has at most one degenerate eigenvalue.

Current Research

1. **Compactness of $\bar{\partial}$-Neumann operator, Property(P_q), and Absence of Analytic Discs.** On locally convexifiable domains compactness of the $\bar{\partial}$-Neumann operator, property(P_q), and absence of analytic discs in the $b\Omega$ are all equivalent. If the boundary of a domain contains an analytic disc, then property(P_q) fails, and compactness is known to fail when the domain is in $\mathbb{C}^2$ and has at least Lipschitz boundary (some boundary regularity is necessary, see [5]). However absence of discs is not sufficient for property(P_q) or for compactness ([9], [7]) to hold. Since the proof of Straube's theorem and it's extension to $\mathbb{C}^n, n > 2$, are independent of property$(\tilde{P}_q)$, I would like to investigate whether this theorem could be used to help find the relationships between these three properties. I am curious to know whether this theorem could be

used to give an example of a domain in $\mathbb{C}^n$ for $n \geq 3$ which has an analytic disc in the boundary and compact $\bar{\partial}$-Neumann operator. By what was said above, the domain would have to fail to be locally convexifiable. One approach is to try to put a Kohn-Nirenberg type domain as a section in a direction transverse to the disc. The difficulty with this example so far has been in finding a smoothly varying real subspace which is in the eigenspace of the smallest eigenvalue of the Levi form. This raises the question whether the hypothesis of the theorem excludes discs from the boundary when $n \geq 3$ (it trivially excludes discs when $n = 2$).

The obvious examples that satisfy the assumptions of the theorem also satisfy property($\tilde{P}_q$). Therefore whether these geometric conditions imply property($\tilde{P}_q$) is not known at present. Since the existence of analytic discs in the boundary is enough for property($\tilde{P}_q$) to fail, answers to the questions mentioned in the previous paragraphs will clarify the relationship between property($\tilde{P}_q$) and compactness of $\bar{\partial}$-Neumann operator as well.

2. **Compactness of Complex Green Operator.** The classical $\bar{\partial}$ problem for domains have an analogous formulation for boundary of domains. This problem is known as the $\bar{\partial}_b$ problem. For the $\bar{\partial}_b$ problem there is a natural symmetry occuring between q forms and $(n-q-1)$ forms; for this reason the regularity properties of $\bar{\partial}_b$ differ from regularity properties of $\bar{\partial}$. In [8] Straube and Raich studied the relationship between compactness of the complex Green operator G_q (for $\bar{\partial}_b$) and the compactness of the $\bar{\partial}$-Neumann operator N_q (for $\bar{\partial}$). It was shown there that smooth domains that satisfy properties (P_q) and (P_{n-1-q}) (see introduction) have compact complex Green operators G_q and G_{n-1-q}. In a joint project with Straube, we are are currently investigating whether geometric conditions from above are sufficient for compactness of the complex Green operators as well.

3. L^p **Regularity of Integral Operators.** In 1978 Kerzman and Stein [6] studied boundary L^p regularity of a Cauchy-type integral operator, associated with the Henkin-Ramirez generating form, acting on functions (that is "0 forms") defined on the boundary of a C^3-smooth strictly pseudoconvex domain in $\mathbb{C}^n$. In a recent joint project with Lanzani, we are investigating an analogous operator now acting on $(0,q)$-forms for $1 \leq q \leq n$.

References

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