GLOBAL REGULARITY FOR THE $\bar{\partial}$-NEUMANN OPERATOR AND BOUNDED PLURISUBHARMONIC EXHAUSTION FUNCTIONS

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Abstract. For a smooth, bounded, pseudoconvex domain $\Omega \subset \mathbb{C}^n$, we derive a new sufficient condition for global regularity of the $\bar{\partial}$-Neumann operator that generalizes McNeal's Property $(\tilde{P})$, the approximately holomorphic vector fields of Boas and Straube, and a condition involving bounded plurisubharmonic exhaustion functions due to Kohn.

1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain. If $\bar{\partial}$ denotes the L^2-closure of the Cauchy-Riemann operator and $\bar{\partial}^*$ its Hilbert space adjoint with respect to L^2 (with the induced boundary condition on $\mathrm{Dom}(\bar{\partial}^*)$), then we can define a self-adjoint operator $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. The $\bar{\partial}$-Neumann Problem is to find a solution $u \in \mathrm{Dom}(\Box)$ to the equation $\Box u = f$ for any $f \in L^2_{(p,q)}(\Omega)$. By a result of Hörmander [18], the solution operator $N : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ exists on all bounded pseudoconvex domains (even without a smoothness assumption on Ω). The $\bar{\partial}$-Neumann operator is said to be globally regular when it preserves the space of forms that are smooth up to the boundary, i.e. $N : C^\infty_{(p,q)}(\bar{\Omega}) \to C^\infty_{(p,q)}(\bar{\Omega})$. Our goal in this paper is to unify three known sufficient conditions for global regularity.

The $\bar{\partial}$-Neumann problem was first solved on smooth, strictly pseudoconvex domains by Kohn in [19, 20] (see [5], [8], or [13] for background). Using work of Barrett [1], Christ [9] was able to show that the $\bar{\partial}$-Neumann operator is not globally regular on the worm domains of Diederich and Fornaess [11]. Several approaches have been taken to finding sufficient conditions for global regularity.

In [22], Kohn and Nirenberg showed that the $\bar{\partial}$-Neumann operator is globally regular if it is compact. However, finding sufficient conditions for compactness is a significant problem its own right (see [15]). One important sufficient condition is Catlin's Property (P) [7], later generalized by McNeal [23] to Property $(\tilde{P})$. Both of these conditions involve families of plurisubharmonic functions λ_M on Ω satisfying the estimate

$$i\partial\bar{\partial}\lambda_M \geq iM\partial\bar{\partial}\,|z|^2$$

on the boundary of Ω for all $M > 0$, in addition to a uniformity condition (uniform boundedness for (P), self-bounded gradient for $(\tilde{P})$; see Definition 2.3).

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A more geometric approach was introduced by Boas and Straube in [4]. They consider $(1,0)$ vector fields that are transverse to the boundary. A family of such vector fields v_ε is called approximately holomorphic if

$$\left|d\rho([\bar\partial, v_\varepsilon])\right| < \varepsilon$$

at points of infinite type on the boundary of Ω for all $\varepsilon > 0$ where ρ is a smooth defining function for Ω. In addition, the vector fields are required to satisfy uniform upper and lower bounds on $|d\rho(v_\varepsilon)|$ and the reality condition $|\arg d\rho(v_\varepsilon)| < \varepsilon$. Under these conditions, Boas and Straube show that the $\bar\partial$-Neumann operator is globally regular.

Our method in this paper is to combine both of these approaches: we look for plurisubharmonic functions λ_ε with self-bounded gradient and transverse $(1,0)$-vector fields v_ε satisfying $|\arg d\rho(v_\varepsilon)| < \varepsilon$ such that

$$i\partial\bar\partial\lambda_\varepsilon \geq i\frac{1}{\varepsilon^2}\overline{\frac{d\rho([\bar\partial, v_\varepsilon])}{d\rho(v_\varepsilon)}} \wedge \frac{d\rho([\bar\partial, v_\varepsilon])}{d\rho(v_\varepsilon)}$$

on the boundary of Ω for all $\varepsilon > 0$. Our uniformity condition on v_ε is relaxed to the requirement that $\frac{d\rho([\bar\partial, v_\varepsilon])}{d\rho(v_\varepsilon)}$ is $o(1/\varepsilon)$ on the boundary of Ω. An alternative approach to unifying these two methods was introduced by Straube in [26]. His approach is *a priori* more general in that it includes compactness without assuming Property $(\tilde P)$. However, our approach is able to incorporate a third condition.

In [21], Kohn considered defining functions ρ such that $-(-\rho)^\eta$ is plurisubharmonic for some constant $0 < \eta < 1$. Such functions are known to exist on any bounded pseudoconvex domain with a C^2 boundary by a result of Diederich and Fornaess [12], but the value of η may need to be very close to 0 (Range has a simplified proof in case the boundary is C^3 [24]). Kohn showed that if the value of η can be taken arbitrarily close to 1 for some family of functions satisfying an appropriate uniformity condition, then the $\bar\partial$-Neumann operator is globally regular. A related condition was used by Herbig and McNeal in [17]. We will show that Kohn's condition also implies our condition.

In Section 2 we will introduce our key definitions, including our sufficient condition for global regularity. Section 3 will establish some basic estimates and Section 4 will set up the key computations for estimating Sobolev norms with these estimates. Finally, Section 5 will prove our main global regularity result and Section 6 will show the connection between Kohn's result and our sufficient condition.

2. Definitions

Let $\mathbb{C}^n$ be endowed with the real Euclidean metric (i.e. $|dz_j|^2 = 2$). This convention dictates that, for example, $|\partial\rho|^2 = 2\sum_{j=1}^n \left|\frac{\partial\rho}{\partial z_j}\right|^2$. Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain, with $\partial\Omega$ denoting the boundary of Ω. Recall that Ω is smooth if there exists a smooth function ρ satisfying $\Omega = \{z : \rho(z) < 0\}$ and $|d\rho| \neq 0$ on $\partial\Omega$ (such a ρ is called a smooth defining function for Ω). Pseudoconvexity means that

$$\sum_{j,k=1}^n a^j \frac{\partial^2\rho}{\partial z_j \partial\bar z_k}\bar a^k \geq 0$$

on $\partial\Omega$ for all $(1,0)$-vector fields a satisfying

$$\sum_{j=1}^{n} a^j \frac{\partial \rho}{\partial z_j} = 0$$

on $\partial\Omega$ (such vector fields are called tangential).

For the L^2 inner product and associated norm, we use the notation $(f,g) = \int_\Omega \langle f,g\rangle\, dV$ and $\|f\|^2 = (f,f)$. Similarly, for a weight function φ, we define the weighted inner product and norm $(f,g)_\varphi = \int_\Omega e^{-\varphi}\langle f,g\rangle\, dV$ and $\|f\|_\varphi^2 = (f,f)_\varphi$. When dealing with differential forms, we let $\mathcal{I}_q$ denote the set of all increasing multi-indices of length q. Hence, a $(0,q)$-form f can be written

$$f = \sum_{J\in\mathcal{I}_q} f_J d\bar{z}_J.$$

When it becomes necessary to use f_J for multi-indices J that are not necessarily increasing, we assume that f_J is skew-symmetric with respect to indices. We use $\bar\partial$ to denote the Cauchy-Riemann operator

$$\bar\partial\left(\sum_{J\in\mathcal{I}_q} f_J d\bar{z}_J\right) = \sum_{j=1}^{n}\sum_{J\in\mathcal{I}_q} \frac{\partial}{\partial \bar{z}_j} f_J d\bar{z}_j \wedge d\bar{z}_J$$

and ϑ to denote the formal adjoint

$$\vartheta\left(\sum_{J\in\mathcal{I}_q} f_J d\bar{z}_J\right) = -\sum_{j=1}^{n}\sum_{I\in\mathcal{I}_{q-1}} \frac{\partial}{\partial z_j} f_{jI} d\bar{z}_I.$$

The adjoint of $\bar\partial$ with respect to the L^2 inner product is denoted $\bar\partial^*$, or $\bar\partial^*_\varphi$ with respect to the weighted inner product. In each case, $\mathrm{Dom}(\bar\partial^*)$ has an associated boundary condition

$$\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j} f_{jI} = 0$$

for all $I \in \mathcal{I}_{q-1}$ and any defining function ρ for Ω.

We use $P : L^2_{(0,q)}(\Omega) \to \ker\bar\partial$ to denote the orthogonal projection onto the space of $\bar\partial$-close forms, also known as the Bergman Projection. If this projection is taken with respect to the weighted inner product $(f,g)_\varphi$, we use the notation P_φ.

Definition 2.1. Let $\Omega \subset \mathbb{C}^n$ be a smooth domain with a smooth defining function ρ. Given a smooth $(1,0)$-vector field v satisfying $d\rho(v) \neq 0$ on $\partial\Omega$, we define the transverse holomorphicity of v (with respect to ρ) to be the $(0,1)$-form

$$\theta^v = \frac{d\rho([\bar\partial, v])}{d\rho(v)}.$$

Remark 2.2. The normalization in the definition of θ^v ensures that θ^v is independent of the choice of ρ on $\partial\Omega$, justifying our notation. The behavior of θ^v will depend on ρ off the boundary, but this effect is negligible for our computations.

To clarify the notation used in computations, we collect some basic expressions in coordinates

$$d\rho(v) = v\rho = \sum_{j=1}^{n} v^j \frac{\partial \rho}{\partial z_j}$$

$$\theta^v = \sum_{j=1}^{n} \theta_j^v d\bar{z}_j = (v\rho)^{-1} \sum_{j,k=1}^{n} \frac{\partial \rho}{\partial z_k} \frac{\partial v^k}{\partial \bar{z}_j} d\bar{z}_j.$$

Definition 2.3. Given a smooth plurisubharmonic function λ on $\bar{\Omega}$, we define a pointwise norm for $(0,1)$-forms θ by

$$|\theta|_{i\partial\bar{\partial}\lambda} = \inf\left\{\varepsilon \in \mathbb{R} : i\varepsilon^2 \partial\bar{\partial}\lambda \geq i\bar{\theta} \wedge \theta\right\}.$$

We say that λ has a self-bounded gradient if $|\bar{\partial}\lambda|_{i\partial\bar{\partial}\lambda} \leq 1$ on Ω.

Remark 2.4. The definition of self-bounded gradient comes from [23]. As in McNeal's paper, we note that a plurisubharmonic function satisfying the uniform bound $0 \leq \lambda_0 \leq 1$ can be turned into a function with a self-bounded gradient via the transformation $\lambda = e^{\lambda_0}$. Such a transformation will also preserve the properties of the complex hessian used in the following definition, and hence the following definition can be strengthened by replacing the self-bounded gradient requirement with a uniform bound. This will yield results closer to those in Catlin [7]. However, such a definition is apparently too strong for the application in Section 6.

Definition 2.5. Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain. We say Ω has a family of transverse vector fields satisfying Property $(\tilde{P})$ if for every $\varepsilon > 0$ there exists a smooth $(1,0)$-vector field v_ε satisfying $|\arg d\rho(v_\varepsilon)| \leq \varepsilon$ and $|\theta^{v_\varepsilon}| \leq \frac{A_\varepsilon}{\varepsilon}$ on $\partial\Omega$ for some family of constants A_ε satisfying $A_\varepsilon \to 0$ and a smooth plurisubharmonic function λ_ε with self-bounded gradient such that $|\theta^{v_\varepsilon}|_{i\partial\bar{\partial}\lambda_\varepsilon} \leq \varepsilon$ on $\partial\Omega$.

In practice, the following form of Definition 2.5 will be more helpful.

Lemma 2.6. *Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain with a smooth defining function ρ and a family of transverse vector fields satisfying Definition 2.5. Then for every $0 < \gamma < 1$ and $\varepsilon > 0$ there exists a neighborhood U_ε of $\partial\Omega$, a smooth $(1,0)$ vector field v_ε satisfying $|\arg d\rho(v_\varepsilon)| < \varepsilon$ and $|\theta^{v_\varepsilon}| < \frac{B_\varepsilon}{\varepsilon}$ on U_ε for some family of constants B_ε satisfying $B_\varepsilon \to 0$ and a smooth plurisubharmonic function λ_ε with self-bounded gradient such that $|\theta^{v_\varepsilon}|_{i\partial\bar{\partial}\lambda_\varepsilon} < \varepsilon$ on U_ε and $i\partial\bar{\partial}\lambda_\varepsilon \geq i\frac{\gamma}{d^2}\partial\bar{\partial}|z|^2$ on Ω for $d = \sup_{z\in\Omega}|z|$.*

Proof. Set $\varepsilon_0 = \frac{1}{2}\varepsilon\sqrt{1-\gamma}$ and let v_0 be a smooth $(1,0)$ vector field satisfying $|\arg d\rho(v_0)| \leq \varepsilon_0$ and $|\theta^{v_0}| \leq \frac{A_{\varepsilon_0}}{\varepsilon_0}$ on $\partial\Omega$ and a smooth plurisubharmonic function λ_0 with self-bounded gradient such that $|\theta^{v_0}|_{i\partial\bar{\partial}\lambda_0} \leq \varepsilon_0$ on $\partial\Omega$. Let

$$\lambda_\varepsilon = (1-\gamma)\lambda_0 + \gamma\frac{|z|^2}{d^2}.$$

We immediately have

$$i\partial\bar{\partial}\lambda_\varepsilon \geq i\frac{\gamma}{d^2}\partial\bar{\partial}|z|^2$$

on $\bar{\Omega}$ and

$$i\partial\bar{\partial}\lambda_\varepsilon \geq i(1-\gamma)\frac{1}{\varepsilon_0^2}\bar{\theta}^{v_0} \wedge \theta^{v_0} > i\frac{1}{\varepsilon^2}\bar{\theta}^{v_0} \wedge \theta^{v_0}$$

on $\partial\Omega$. If we set $v_\varepsilon = \frac{1}{\sqrt{1-\gamma}} v_0$ (note $\theta^{v_\varepsilon} = \theta^{v_0}$) and $B_\varepsilon = \frac{3A_{\varepsilon_0}}{\sqrt{1-\gamma}}$, we have $|\theta^{v_\varepsilon}|_{i\partial\bar\partial\lambda_\varepsilon} < \varepsilon$, $|\theta^{v_\varepsilon}| < \frac{B_\varepsilon}{\varepsilon}$ and $|\arg(d\rho(v_\varepsilon))| < \varepsilon$ on $\partial\Omega$. Since these are open conditions, there exists a neighborhood U_ε of $\partial\Omega$ on which they still hold. It remains to see that $|\bar\partial\lambda_\varepsilon|^2_{i\partial\bar\partial\lambda_\varepsilon} \le 1$. We compute

$$\begin{aligned} i\partial\bar\partial\lambda_\varepsilon - i\partial\lambda_\varepsilon \wedge \bar\partial\lambda_\varepsilon &= i(1-\gamma)\partial\bar\partial\lambda_0 + i\frac{\gamma}{d^2}\partial\bar\partial\,|z|^2 \\ &\qquad - i\left((1-\gamma)\partial\lambda_0 + \frac{\gamma}{d^2}\partial\,|z|^2\right) \wedge \left((1-\gamma)\bar\partial\lambda_0 + \frac{\gamma}{d^2}\bar\partial\,|z|^2\right) \\ &\ge i(1-\gamma)\partial\lambda_0 \wedge \bar\partial\lambda_0 + i\frac{\gamma}{d^4}\partial\,|z|^2 \wedge \bar\partial\,|z|^2 \\ &\qquad - i\left((1-\gamma)\partial\lambda_0 + \frac{\gamma}{d^2}\partial\,|z|^2\right) \wedge \left((1-\gamma)\bar\partial\lambda_0 + \frac{\gamma}{d^2}\bar\partial\,|z|^2\right) \\ &= i\gamma(1-\gamma)\partial\lambda_0 \wedge \bar\partial\lambda_0 - i\frac{\gamma(1-\gamma)}{d^2}\left(\partial\lambda_0 \wedge \bar\partial\,|z|^2 + \partial\,|z|^2 \wedge \bar\partial\lambda_0\right) \\ &\qquad + i\frac{\gamma(1-\gamma)}{d^4}\partial\,|z|^2 \wedge \bar\partial\,|z|^2 . \end{aligned}$$

Since the final line is positive, we have $|\bar\partial\lambda_\varepsilon|_{i\partial\bar\partial\lambda_\varepsilon} \le 1$ and the proof is complete. □

Definition 2.5 generalizes several known conditions for global regularity. In [23], McNeal defined Property $(\tilde P)$ (a generalization of Catlin's Property (P) [7]) and showed that it implies compactness for the $\bar\partial$-Neumann operator. This, in turn, implies global regularity by a classic result of Kohn and Nirenberg [22]. In the notation of this paper, Property $(\tilde P)$ is equivalent to the statement that for every $\varepsilon > 0$ there exists a smooth plurisubharmonic function λ_ε with self-bounded gradient such that $|\theta|_{i\partial\bar\partial\lambda_\varepsilon} < \varepsilon\,|\theta|$ on $\partial\Omega$ for any $(0,1)$-form θ. Hence, any fixed transverse vector field v will satisfy Definition 2.5.

Another approach to global regularity was pioneered by Boas and Straube in [4] using approximately holomorphic vector fields. In the most general definition (see [5] and the references therein), a domain Ω possesses a family of approximately holomorphic transverse vector fields if there exists a constant $C > 1$ such that for every $\varepsilon > 0$ there is a vector field v_ε satisfying $\frac{1}{C} < |d\rho(v_\varepsilon)| < C$, $|\arg d\rho(v_\varepsilon)| < \varepsilon$, and $d\rho([\bar\partial, v_\varepsilon]) < \varepsilon$ on the set of points of infinite type in $\partial\Omega$ (for further discussion of finite and infinite type, see [10]).

Proposition 2.7. *Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain that possesses a family of approximately holomorphic transverse vector fields. Then Ω has a family of transverse vector fields satisfying Definition 2.5.*

Proof. Fix a smooth defining function ρ and let $K \subset \partial\Omega$ be the set of points of infinite type. By a result of D'Angelo [10], K is compact. Let $C > 1$ be a constant such that for every $\delta > 0$, there exists a vector field v_δ in a neighborhood of K such that $\frac{1}{C} < |d\rho(v_\delta)| < C$, $|\arg d\rho(v_\delta)| < \delta$, and $|d\rho([\bar\partial, v_\delta])| < \delta$ on K. Since $\partial\bar\partial\,|z|^2$ is twice the Kähler form for the Euclidean metric, this is equivalent to writing $|\theta^{v_\delta}|^2_{i\partial\bar\partial|z|^2} < \frac{1}{2}C^2\delta^2$. To bound our weight function, we let $d = \sup_\Omega |z|$ and define $\phi = \frac{1}{3}\left(\frac{|z|^2}{d^2} + 1\right)$, so that $\frac{1}{3} \le \phi \le \frac{2}{3}$. Then $|\theta^{v_\delta}|^2_{i\partial\bar\partial\phi} < \frac{3}{2}d^2C^2\delta^2$.

Since these inequalities are all strict (and ϕ is strictly plurisubharmonic), they also hold in an open neighborhood U_δ of K. Let U'_δ be an open neighborhood of K such that $\bar U'_\delta \subset U_\delta$ and choose $\chi_\delta \in C^\infty_0(U_\delta)$ such that $\chi_\delta \equiv 1$ on U'_δ. For a given

$\varepsilon > 0$, we set $\delta = \min\left\{\frac{\sqrt{2}\varepsilon}{\sqrt{3}dC}, \varepsilon\right\}$ and define

$$v_\varepsilon = \chi_\delta v_\delta + (1-\chi_\delta) 2\left|\bar\partial\rho\right|^{-2}\sum_{j=1}^n \frac{\partial\rho}{\partial\bar z_j}\frac{\partial}{\partial z_j}.$$

Then on all of $\partial\Omega$, we have $\frac{1}{C} < |d\rho(v_\varepsilon)| < C$ and $|\arg d\rho(v_\varepsilon)| < \varepsilon$, while on U'_δ we have $|\theta^{v_\varepsilon}|_{i\partial\bar\partial\phi} < \varepsilon$.

Now any $p \in \partial\Omega/K$ is a finite type point, so by a result of Catlin [6] there exists an open ball $B(p, r_p)$ such that for any $M > 0$ there is a smooth plurisubharmonic function $\varphi_{p,M}$ on $B(p,r_p)\cap\bar\Omega$ satisfying $0 \le \varphi_{p,M} \le 1$ on $B(p,r_p)\cap\bar\Omega$ and $i\partial\bar\partial\varphi_{p,M} > iM\partial\bar\partial|z|^2$ on $B(p,r_p)\cap\partial\Omega$. Let $\chi_p \in C_0^\infty(B(p,r_p))$ such that $\chi_p \equiv 1$ on $B(p, r_p/2)$ and set $\lambda_{p,M} = \frac{1}{3}(\varphi_{p,M} + 2\chi_p)$. Note that we have $0 \le \lambda_{p,M} \le 1$ and $i\partial\bar\partial\lambda_{p,M} > i\left(\frac{1}{3}M - k_p\right)\partial\bar\partial|z|^2$ on $\partial\Omega$ for some constant $k_p > 0$.

Since $\partial\Omega/U'_\delta$ is compact, we can choose a finite set $\mathcal{P} \subset \partial\Omega/U'_\delta$ such that $\{B(p,r_p/2)\}_{\mathcal{P}}$ covers $\partial\Omega/U'_\delta$. Note that on $\partial B(p,r_p)$, where $\chi_p = 0$, $\lambda_{p,M} \le \frac{1}{3}$, while on $B(p,r_p/2)$, where $\chi_p = 1$, $\lambda_{p,M} \ge \frac{2}{3}$. Hence, we can define

$$\lambda_M(z) = \begin{cases} \sup_{\{p\in\mathcal{P}: z\in B(p,r_p)\}} \lambda_{p,M}(z) & z \notin U'_\delta \\ \max\left\{\sup_{\{p\in\mathcal{P}: z\in B(p,r_p)\}} \lambda_{p,M}(z), \phi(z)\right\} & z \in U'_\delta \end{cases}$$

and obtain a continuous function satisfying $0 \le \lambda_M \le 1$ and on a neighborhood of each boundary point either $i\partial\bar\partial\lambda_M > i\left(\frac{1}{3}M - k\right)\partial\bar\partial|z|^2$ in the sense of currents for some constant $k > 0$ or $|\theta^{v_\varepsilon}|_{i\partial\bar\partial\lambda_M} < \varepsilon$ in the sense of currents. Since k and v_ε do not depend on M, we can choose M sufficiently large so that $|\theta^{v_\varepsilon}|_{i\partial\bar\partial\lambda_M} < \varepsilon$ in the sense of currents on a neighborhood of $\partial\Omega$. On this neighborhood we can use a standard regularization argument involving convolution with a smooth compactly supported function to obtain a smooth function $\tilde\lambda_M$. As noted in [23], a bounded function can always be replaced by a function with self-bounded gradient without affecting the good lower bound on the complex hessian by setting $\lambda_\varepsilon = e^{\tilde\lambda_M}$. By the proof of Lemma 3 in [25], λ_ε can be extended into the interior of Ω without sacrificing the self-bounded gradient. □

3. The Basic Estimate

We begin with the following basic estimate (see [5] for the history and references for this particular estimate).

Proposition 3.1. *Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain with weight functions $\varphi, a \in C^2(\bar\Omega)$, $a \ge 0$. Let $u \in \mathrm{Dom}(\bar\partial^*_\varphi) \cap C^1_{(0,q)}(\bar\Omega)$. We have*

$$\left\|\sqrt{a}\bar\partial u\right\|^2_\varphi + \left\|\sqrt{a}\bar\partial^*_\varphi u\right\|^2_\varphi \ge \left\|\sqrt{a}\bar\nabla u\right\|^2_\varphi + 2\,\mathrm{Re}\left(\bar\partial a \wedge \bar\partial^*_\varphi u, u\right)_\varphi + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^n\left(\left(a\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k} - \frac{\partial^2 a}{\partial z_j\partial\bar z_k}\right)u_{jI}d\bar z_I, u_{kI}d\bar z_I\right)_\varphi. \tag{3.1}$$

We will also need some relationships between estimates for the basic operators.

Lemma 3.2. *Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain, let $\varphi, \psi \in C^\infty(\bar\Omega)$, let $0 < r < 1$ be a constant, and let $M_q : L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$ be defined* *by*

$$M_q\left(\sum_{J\in\mathcal{I}_q} f_J d\bar{z}_J\right) = \sum_{j,k=1}^n \sum_{I\in\mathcal{I}_{q-1}} A_{jk} f_{jI} d\bar{z}_k \wedge d\bar{z}_I$$

for some $1 \leq q \leq n$ *where* A_{jk} *is a continuous family of positive definite hermitian matrices such that for every* $f \in L^2_{(0,q)}(\Omega) \cap \ker\bar\partial \cap \mathrm{Dom}(\bar\partial^*_\varphi)$ *we have*

$$\frac{1}{r}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n \frac{\partial(\varphi-\psi)}{\partial z_j} f_{jI} d\bar{z}_I\right\|_{\varphi+\psi} \leq \sqrt{(M_q f, f)_{\varphi+\psi}} \leq \left\|\bar\partial^*_\varphi f\right\|_{\varphi+\psi}. \tag{3.2}$$

We then have for all $u \in L^2_{(0,q-1)}(\Omega) \cap (\ker\bar\partial)^\perp$

$$\|u\|_{\psi-\varphi} \leq \frac{1}{1-r}\sqrt{(M_q^{-1}\bar\partial u, \bar\partial u)_{\psi-\varphi}} \tag{3.3}$$

and for all $u \in L^2_{(0,q-1)}(\Omega)$

$$\|Pu\|_{\psi-\varphi} \leq \frac{1}{1-r}\|u\|_{\psi-\varphi}. \tag{3.4}$$

Remark 3.3. This result is similar to Theorem 2.3 in [2], but u is in a different domain and we will need the precise computation of the constant C_r.

Proof. Note that M_q is a continuous, self-adjoint, positive definite linear operator, so M_q is invertible and a positive-definite square root of M_q exists. For any $f, g \in L^2_{(0,q)}(\Omega)$, we have

$$|(f,g)_\varphi| = \left|(M_q^{-1/2} f, M_q^{1/2} g)_\varphi\right| \leq \sqrt{(M_q^{-1} f, f)_{\varphi-\psi}(M_q g, g)_{\varphi+\psi}}.$$

If $g \in \ker\bar\partial \cap \mathrm{Dom}(\bar\partial^*_\varphi)$, (3.2) implies

$$|(f,g)_\varphi| \leq \sqrt{(M_q^{-1} f, f)_{\varphi-\psi}}\left\|\bar\partial^*_\varphi g\right\|_{\varphi+\psi}.$$

As in Hörmander [18] we now assume that $f \in \ker\bar\partial$ and decompose $g \in \mathrm{Dom}(\bar\partial^*_\varphi)$ as $g = g_0 + g_1$ where $g_0 \in \ker\bar\partial$ and $g_1 \in (\ker\bar\partial)^\perp_\varphi$. Since $\bar\partial^*_\varphi g = \bar\partial^*_\varphi g_0$ and $(f,g)_\varphi = (f,g_0)_\varphi$, we have the same estimate as above. This implies that $\bar\partial^*_\varphi g \to (f,g)_\varphi$ is a bounded linear functional on $\mathrm{Range}(\bar\partial^*_\varphi)$ in the norm $\|\cdot\|_{\varphi+\psi}$, so there exists $u_0 \in L^2_{(0,q-1)}(\Omega) \cap \mathrm{Range}(\bar\partial^*_\varphi)$ such that $(f,g)_\varphi = (u_0, \bar\partial^*_\varphi g)_{\varphi+\psi}$ for all $g \in \mathrm{Dom}(\bar\partial^*_\varphi)$ and

$$\|u_0\|_{\varphi+\psi} \leq \sqrt{(M_q^{-1} f, f)_{\varphi-\psi}}.$$

Since $(u_0, \bar\partial^*_\varphi g)_{\varphi+\psi} = (\bar\partial(e^{-\psi}u_0), g)_\varphi$, we have $\bar\partial(e^{-\psi}u_0) = f$. Since $u_0 \in \mathrm{Range}(\bar\partial^*_\varphi)$ and $\mathrm{Range}(\bar\partial^*_\varphi) = (\ker\bar\partial)^\perp_\varphi$, $e^{-\varphi}u_0 \in (\ker\bar\partial)^\perp$, so let $u = e^{-\varphi}u_0$. This implies $f = \bar\partial(e^{\varphi-\psi}u) = e^{\varphi-\psi}(\bar\partial u + \bar\partial(\varphi-\psi)\wedge u)$. Substituting, we obtain

$$\|u\|_{\psi-\varphi} \leq \sqrt{(M_q^{-1}(\bar\partial u + \bar\partial(\varphi-\psi)\wedge u), (\bar\partial u + \bar\partial(\varphi-\psi)\wedge u))_{\psi-\varphi}}.$$

Note that

$$\sqrt{(M_q^{-1}\bar\partial(\varphi-\psi)\wedge u, \bar\partial(\varphi-\psi)\wedge u)_{\psi-\varphi}} = \sup_{f\in L^2_{(0,q)}(\Omega), f\neq 0}\frac{\left|(\bar\partial(\varphi-\psi)\wedge u, f)_\psi\right|}{\sqrt{(M_q f, f)_{\varphi+\psi}}}.$$

Applying Cauchy-Schwarz to the numerator followed by (3.2), we have

$$\sqrt{(M_q^{-1}\bar\partial(\varphi-\psi)\wedge u,\bar\partial(\varphi-\psi)\wedge u)_{\psi-\varphi}} \le \sup_{f\in L^2_{(0,q)}(\Omega),f\neq 0}\frac{\|u\|_{\psi-\varphi}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n\frac{\partial(\varphi-\psi)}{\partial z_j}f_{jI}d\bar z_I\right\|_{\varphi+\psi}}{\sqrt{(M_q f,f)_{\varphi+\psi}}}\le r\|u\|_{\psi-\varphi}. \tag{3.5}$$

The triangle inequality with respect to the norm $\sqrt{(M_q^{-1}\cdot,\cdot)_{\psi-\varphi}}$ yields

$$\sqrt{(M_q^{-1}(\bar\partial u+\bar\partial(\varphi-\psi)\wedge u),(\bar\partial u+\bar\partial(\varphi-\psi)\wedge u))_{\psi-\varphi}} \le \sqrt{(M_q^{-1}\bar\partial u,\bar\partial u)_{\psi-\varphi}}+\sqrt{(M_q^{-1}\bar\partial(\varphi-\psi)\wedge u,\bar\partial(\varphi-\psi)\wedge u)_{\psi-\varphi}}\le \sqrt{(M_q^{-1}\bar\partial u,\bar\partial u)_{\psi-\varphi}}+r\|u\|_{\psi-\varphi}$$

Since we now have

$$\|u\|_{\psi-\varphi}\le\sqrt{(M_q^{-1}\bar\partial u,\bar\partial u)_{\psi-\varphi}}+r\|u\|_{\psi-\varphi},$$

we can subtract the final term on the right-hand side from the left-hand side and divide by the resulting coefficient to obtain (3.3).

As in [3], we observe that the unweighted Bergman Projection P can be written in terms of the weighted Bergman Projection $P_{\varphi-\psi}$ for the inner product $(\cdot,\cdot)_{\varphi-\psi}$ as follows

$$Pu=P(e^{\psi-\varphi}P_{\varphi-\psi}(e^{\varphi-\psi}u)).$$

By Kohn's formula $P=I-\bar\partial^*N\bar\partial$, we have

$$Pu=e^{\psi-\varphi}P_{\varphi-\psi}(e^{\varphi-\psi}u)-\bar\partial^*N(e^{\psi-\varphi}\bar\partial(\psi-\varphi)\wedge P_{\varphi-\psi}(e^{\varphi-\psi}u)).$$

Estimating the first term gives us

$$\left\|e^{\psi-\varphi}P_{\varphi-\psi}(e^{\varphi-\psi}u)\right\|_{\psi-\varphi}=\left\|P_{\varphi-\psi}(e^{\varphi-\psi}u)\right\|_{\varphi-\psi}\le\left\|e^{\varphi-\psi}u\right\|_{\varphi-\psi}=\|u\|_{\psi-\varphi}$$

Substituting the second term into (3.3) gives us

$$\left\|\bar\partial^*N(e^{\psi-\varphi}\bar\partial(\psi-\varphi)\wedge P_{\varphi-\psi}(e^{\varphi-\psi}u))\right\|_{\psi-\varphi}\le \frac{1}{1-r}\sqrt{(M_q^{-1}\bar\partial(\psi-\varphi)\wedge P_{\varphi-\psi}(e^{\varphi-\psi}u),\bar\partial(\psi-\varphi)\wedge P_{\varphi-\psi}(e^{\varphi-\psi}u))_{\varphi-\psi}},$$

so by (3.5)

$$\left\|\bar\partial^*N(e^{\psi-\varphi}\bar\partial(\psi-\varphi)\wedge P_{\varphi-\psi}(e^{\varphi-\psi}u))\right\|_{\psi-\varphi}\le\frac{r}{1-r}\|u\|_{\psi-\varphi}.$$

Combining these, we have

$$\begin{aligned}\|Pu\|_{\psi-\varphi}&\le\left\|e^{\psi-\varphi}P_{\varphi-\psi}(e^{\varphi-\psi}u)\right\|_{\psi-\varphi}\\&\quad+\left\|\bar\partial^*N(e^{\psi-\varphi}\bar\partial(\psi-\varphi)\wedge P_{\varphi-\psi}(e^{\varphi-\psi}u))\right\|_{\psi-\varphi}\\&\le\left(1+\frac{r}{1-r}\right)\|u\|_{\psi-\varphi}.\end{aligned}$$

If we simplify, (3.4) follows. □

From this, we will derive our basic estimate.

Proposition 3.4. *Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain with a smooth defining function ρ and a family of transverse vector fields satisfying the conditions of Lemma 2.6. Then for any $\varepsilon > 0$ and $0 < t < \frac{1}{3}$ there exists a $(0,1)$-form θ_ε such that $\theta_\varepsilon = \theta^{v_\varepsilon}$ near $\partial\Omega$, a constant $C_t > 0$, and a norm $\|\cdot\|_{t,\varepsilon}$ defined by (3.10) equivalent to the L^2-norm satisfying*

$$\|\theta_\varepsilon \wedge u\|_{t,\varepsilon} \leq \varepsilon C_t \|u\|_{t\lambda_\varepsilon} \text{ and } \|\bar\partial u\|_{t,\varepsilon} \leq C_t \|\bar\partial u\|_{t\lambda_\varepsilon}.$$

for $u \in L^2_{(0,q-1)}(\Omega) \cap \mathrm{Dom}(\bar\partial)$, $1 \leq q \leq n$. Furthermore

1. *For any $u \in L^2_{(0,q-1)}(\Omega) \cap (\ker\bar\partial)^\perp$, $1 \leq q \leq n$, we have*

$$\|u\|_{t\lambda_\varepsilon} \leq \|\bar\partial u\|_{t,\varepsilon}. \tag{3.6}$$

2. *For $u \in L^2_{(0,q-1)}(\Omega)$, $1 \leq q \leq n$ we have*

$$\|Pu\|_{t\lambda_\varepsilon} \leq C_t \|u\|_{t\lambda_\varepsilon}. \tag{3.7}$$

3. *For all $u \in \mathrm{Dom}(\bar\partial^*) \cap \mathrm{Dom}(\bar\partial) \cap L^2_{(0,q)}(\Omega)$, $1 \leq q \leq n$, we have*

$$\left\| \sum_{I \in \mathcal{I}_{q-1}} \sum_{j=1}^n (\bar\theta_\varepsilon)_j u_{jI} d\bar z_I \right\|_{t\lambda_\varepsilon} \leq \varepsilon C_t (\|\bar\partial^* u\|_{t\lambda_\varepsilon} + \|\bar\partial u\|_{t\lambda_\varepsilon}), \tag{3.8}$$

and

$$\|u\|_{t\lambda_\varepsilon} \leq C_t (\|\bar\partial^* u\|_{t\lambda_\varepsilon} + \|\bar\partial u\|_{t\lambda_\varepsilon}). \tag{3.9}$$

Proof. Fix a smooth defining function ρ and let $\varepsilon > 0$ be given. Let U_ε, λ_ε, v_ε and γ be as in Lemma 2.6, and let $\chi_\varepsilon \in C^\infty_0(U_\varepsilon)$ such that $\chi_\varepsilon \equiv 1$ in a neighborhood of $\partial\Omega$. Let $\theta_\varepsilon = \chi_\varepsilon \theta^{v_\varepsilon}$ on U_ε and $\theta_\varepsilon = 0$ otherwise, so that $|\theta_\varepsilon|_{i\partial\bar\partial\lambda_\varepsilon} \leq \varepsilon$ on all of Ω. Let $a = e^{-\frac{1+t}{2}\lambda_\varepsilon}$ and $\varphi = \frac{1-t}{2}\lambda_\varepsilon$. On U_ε, we compute

$$ia\partial\bar\partial\varphi - i\partial\bar\partial a = ia\left(\partial\bar\partial\lambda_\varepsilon - \left(\frac{1+t}{2}\right)^2 \partial\lambda_\varepsilon \wedge \bar\partial\lambda_\varepsilon\right).$$

Define M_q on $L^2_{(0,q)}(\Omega)$ by

$$M_q f = \frac{1-t}{2} \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^n \left(\frac{\partial^2\lambda_\varepsilon}{\partial z_j \partial\bar z_k} - \left(\frac{1+t}{2}\right) \frac{\partial\lambda_\varepsilon}{\partial z_j} \frac{\partial\lambda_\varepsilon}{\partial\bar z_k}\right) f_{jI} d\bar z_k \wedge d\bar z_I$$

Note that the change from $\left(\frac{1+t}{2}\right)^2$ to $\frac{1+t}{2}$ in the previous two equations is not a mistake; the difference between these is responsible for the coefficient $\frac{1-t^2}{4}$ in the following equation. For $f \in \mathrm{Dom}(\bar\partial^*_\varphi) \cap C^1_{(0,q)}(\bar\Omega)$, $1 \leq q \leq n$, we can substitute into (3.1) to obtain

$$\begin{aligned}\|\bar\partial f\|^2_{\lambda_\varepsilon} + \|\bar\partial^*_\varphi f\|^2_{\lambda_\varepsilon} \geq \|\bar\nabla f\|^2_{\lambda_\varepsilon} - (1+t)\,\mathrm{Re}\left(\bar\partial\lambda_\varepsilon \wedge \bar\partial^*_\varphi f, f\right)_{\lambda_\varepsilon} \\ + \frac{2}{1-t}(M_q f, f)_{\lambda_\varepsilon} + \frac{1-t^2}{4}\left\| \sum_{I \in \mathcal{I}_{q-1}} \sum_{j=1}^n \frac{\partial\lambda_\varepsilon}{\partial z_j} f_{jI} d\bar z_I \right\|^2_{\lambda_\varepsilon}.\end{aligned}$$

If we observe that

$$\left|\mathrm{Re}\left(\bar\partial\lambda_\varepsilon \wedge \bar\partial^*_\varphi f, f\right)_{\lambda_\varepsilon}\right| \leq \left\|\bar\partial^*_\varphi f\right\|_{\lambda_\varepsilon} \left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n \frac{\partial\lambda_\varepsilon}{\partial\bar z_j} f_{jI} d\bar z_I\right\|_{\lambda_\varepsilon}$$
$$\leq \frac{1}{1-t}\left\|\bar\partial^*_\varphi f\right\|^2_{\lambda_\varepsilon} + \frac{1-t}{4}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n \frac{\partial\lambda_\varepsilon}{\partial\bar z_j} f_{jI} d\bar z_I\right\|^2_{\lambda_\varepsilon}$$

we can substitute and obtain

$$\left\|\bar\partial f\right\|^2_{\lambda_\varepsilon} + \frac{2}{1-t}\left\|\bar\partial^*_\varphi f\right\|^2_{\lambda_\varepsilon} \geq \left\|\bar\nabla f\right\|^2_{\lambda_\varepsilon} + \frac{2}{1-t}(M_q f, f)_{\lambda_\varepsilon}.$$

By the usual density result (see for example Lemma 4.3.2 in [8]), we obtain that for all $f\in\mathrm{Dom}(\bar\partial^*_\varphi)\cap\mathrm{Dom}(\bar\partial)\cap L^2_{(0,q)}(\Omega)$, $1\leq q\leq n$, we have

$$\left\|\bar\partial f\right\|^2_{\lambda_\varepsilon} + \frac{2}{1-t}\left\|\bar\partial^*_\varphi f\right\|^2_{\lambda_\varepsilon} \geq \frac{2}{1-t}(M_q f, f)_{\lambda_\varepsilon}.$$

For $f\in\ker\bar\partial$, this simplifies to $\left\|\bar\partial^*_\varphi f\right\|_{\lambda_\varepsilon}\geq\sqrt{(M_q f,f)_{\lambda_\varepsilon}}$. To apply Lemma 3.2, we set $\psi=\frac{1+t}{2}\lambda_\varepsilon$ and compute $\bar\partial(\varphi-\psi)=-t\bar\partial\lambda_\varepsilon$. Hence

$$\sqrt{(M_q f, f)_{\lambda_\varepsilon}} \geq \frac{1-t}{2}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n \frac{\partial\lambda_\varepsilon}{\partial\bar z_j} f_{jI} d\bar z_I\right\|_{\lambda_\varepsilon}$$
$$= \frac{1-t}{2t}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n \frac{\partial(\varphi-\psi)}{\partial\bar z_j} f_{jI} d\bar z_I\right\|_{\varphi+\psi},$$

so when $0<t<\frac{1}{3}$ we can take $r=\frac{2t}{1-t}$ and apply Lemma 3.2 to obtain

$$\|u\|_{t\lambda_\varepsilon} \leq \frac{1-t}{1-3t}\sqrt{(M_q^{-1}\bar\partial u, \bar\partial u)_{t\lambda_\varepsilon}}$$

for $u\in L^2_{(0,q-1)}(\Omega)\cap(\ker\bar\partial)^\perp$ and

$$\|Pu\|_{t\lambda_\varepsilon} \leq \frac{1-t}{1-3t}\|u\|_{t\lambda_\varepsilon}$$

for $u\in L^2_{(0,q-1)}(\Omega)$. This immediately gives us (3.7), and we define

$$\|u\|_{t,\varepsilon} = \frac{1-t}{1-3t}\sqrt{(M_q^{-1}\bar\partial u, \bar\partial u)_{t\lambda_\varepsilon}} \tag{3.10}$$

so that (3.6) follows. Since

$$\sqrt{(M_q f, f)} \geq \frac{1-t}{2\varepsilon}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n (\bar\theta_\varepsilon)_j f_{jI} d\bar z_I\right\|$$

we can use a duality argument (as in the proof of Lemma 3.2) to show that

$$\frac{1-3t}{1-t}\|\theta_\varepsilon\wedge u\|_{t,\varepsilon} = \sqrt{(M_q^{-1}\theta_\varepsilon\wedge u, \theta_\varepsilon\wedge u)_{t\lambda_\varepsilon}} \leq \frac{2\varepsilon}{1-t}\|u\|_{t\lambda_\varepsilon}.$$

Similarly,

$$\sqrt{(M_q f, f)} \geq \sqrt{2}\frac{1-t}{2}\frac{\sqrt{\gamma q}}{d}\|f\|$$

(since $\partial\bar{\partial}|z|^2$ is twice the identity for the Euclidean metric) implies

$$\frac{1-3t}{1-t}\left\|\bar{\partial}u\right\|_{t,\varepsilon} = \sqrt{(M_q^{-1}\bar{\partial}u, \bar{\partial}u)_{t\lambda_\varepsilon}} \leq \frac{\sqrt{2}d}{\sqrt{\gamma q}(1-t)}\left\|\bar{\partial}u\right\|_{t\lambda_\varepsilon}.$$

Next, we let $a = e^{-t\lambda_\varepsilon}$ and $\varphi = 0$. Fix $1 > s > t$. On U_ε, we compute

$$\begin{aligned}-i\partial\bar{\partial}a &= ite^{-t\lambda_\varepsilon}\partial\bar{\partial}\lambda_\varepsilon - it^2e^{-t\lambda_\varepsilon}\partial\lambda_\varepsilon\wedge\bar{\partial}\lambda_\varepsilon \\ &\geq ie^{-t\lambda_\varepsilon}\frac{t(1-s)}{\varepsilon^2}\bar{\theta}_\varepsilon\wedge\theta_\varepsilon + it(s-t)e^{-t\lambda_\varepsilon}\partial\lambda_\varepsilon\wedge\bar{\partial}\lambda_\varepsilon.\end{aligned}$$

For $u \in \operatorname{Dom}(\bar{\partial}^*) \cap C^1_{(0,q)}(\bar{\Omega})$, $1 \leq q \leq n$, we can substitute into (3.1) to obtain

$$\begin{aligned}&\left\|\bar{\partial}u\right\|^2_{t\lambda_\varepsilon} + \left\|\bar{\partial}^*u\right\|^2_{t\lambda_\varepsilon} \geq \left\|\bar{\nabla}u\right\|^2_{t\lambda_\varepsilon} - 2t\operatorname{Re}\left(\bar{\partial}\lambda_\varepsilon\wedge\bar{\partial}^*u, u\right)_{t\lambda_\varepsilon} \\ &\quad + \frac{t(1-s)}{\varepsilon^2}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar{\theta}_\varepsilon)_j u_{jI}d\bar{z}_I\right\|^2_{t\lambda_\varepsilon} + t(s-t)\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n\frac{\partial\lambda_\varepsilon}{\partial z_j}u_{jI}d\bar{z}_I\right\|^2_{t\lambda_\varepsilon}.\end{aligned}$$

If we observe that

$$\begin{aligned}\left|2\operatorname{Re}\left(\bar{\partial}\lambda_\varepsilon\wedge\bar{\partial}^*u, u\right)_{t\lambda_\varepsilon}\right| &\leq 2\left\|\bar{\partial}^*u\right\|_{t\lambda_\varepsilon}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n\frac{\partial\lambda_\varepsilon}{\partial\bar{z}_j}u_{jI}d\bar{z}_I\right\|_{t\lambda_\varepsilon} \\ &\leq \frac{1}{s-t}\left\|\bar{\partial}^*u\right\|^2_{t\lambda_\varepsilon} + (s-t)\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n\frac{\partial\lambda_\varepsilon}{\partial\bar{z}_j}u_{jI}d\bar{z}_I\right\|^2_{t\lambda_\varepsilon}\end{aligned}$$

we can substitute and obtain

$$\left\|\bar{\partial}u\right\|^2_{t\lambda_\varepsilon} + \frac{s}{s-t}\left\|\bar{\partial}^*u\right\|^2_{t\lambda_\varepsilon} \geq \left\|\bar{\nabla}u\right\|^2_{t\lambda_\varepsilon} + \frac{t(1-s)}{\varepsilon^2}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar{\theta}_\varepsilon)_j u_{jI}d\bar{z}_I\right\|^2_{t\lambda_\varepsilon}.$$

Using density again, we obtain that for all $u \in \operatorname{Dom}(\bar{\partial}^*_\varphi) \cap \operatorname{Dom}(\bar{\partial}) \cap L^2_{(0,q)}(\Omega)$, $1 \leq q \leq n$, we have

$$\left\|\bar{\partial}u\right\|^2_{t\lambda_\varepsilon} + \frac{s}{s-t}\left\|\bar{\partial}^*u\right\|^2_{t\lambda_\varepsilon} \geq \frac{t(1-s)}{\varepsilon^2}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar{\theta}_\varepsilon)_j u_{jI}d\bar{z}_I\right\|^2_{t\lambda_\varepsilon},$$

and (3.8) follows.

Since λ_ε is strictly plurisubharmonic with self-bounded gradient and uniform lower bound on the complex hessian, we can also use (3.1) with $a = e^{-t\lambda_\varepsilon}$ and $\varphi = 0$ to obtain (3.9). □

4. Estimating Sobolev Norms

Fix a smooth defining function ρ and a constant $0 < t < \frac{1}{3}$. In what follows, we adopt the convention that $C > 0$ is a constant depending entirely on t, derivatives of ρ, and the diameter of Ω, whose value can change from line to line. We will follow a similar convention with $C_\varepsilon > 0$, except that C_ε will also depend on ε. Conventions for C_k and $C_{k,\varepsilon}$ will be similar. Before proceeding with the main proof, we collect several crucial estimates that we will need.

Lemma 4.1. *Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain with smooth defining function ρ, and let $u \in C^\infty_{(0,q)}(\bar{\Omega}) \cap \mathrm{Dom}(\bar{\partial}^*)$, $1 \leq q \leq n$. We have*

(1) *For any $k \in \mathbb{N}$,*

$$\sum_{j=1}^{n} \left\| \frac{\partial u}{\partial \bar{z}_j} \right\|_{k-1}^2 \leq C_k \left(\left\|\bar{\partial} u\right\|_{k-1}^2 + \left\|\bar{\partial}^* u\right\|_{k-1}^2 + \|u\|_{k-1}^2 \right). \tag{4.1}$$

(2) *For any $k \in \mathbb{N}$ and smooth $(1,0)$ vector field Y satisfying $Y\rho = 0$ on $\partial\Omega$*

$$\|Yu\|_{k-1}^2 \leq C_k \left(\left\|\bar{\partial} u\right\|_{k-1}^2 + \left\|\bar{\partial}^* u\right\|_{k-1}^2 + \|u\|_{k-1} \|u\|_k \right). \tag{4.2}$$

(3) *If u is compactly supported in Ω and $k \in \mathbb{N}$*

$$\|u\|_k^2 \leq C_k \left(\left\|\bar{\partial} u\right\|_{k-1}^2 + \left\|\bar{\partial}^* u\right\|_{k-1}^2 + \|u\|_{k-1}^2 \right). \tag{4.3}$$

The first two inequalities are due to Boas and Straube [4], and the third is Gårding's inequality (which follows from the fact that the system $\bar{\partial} \oplus \bar{\partial}^*$ is elliptic in the interior). When $q = 0$, it is trivial that for any $u \in C^\infty(\overline{\Omega})$, and $k \in \mathbb{N}$

$$\sum_{j=1}^{n} \left\| \frac{\partial u}{\partial \bar{z}_j} \right\|_{k-1}^2 = \left\|\bar{\partial} u\right\|_{k-1}^2. \tag{4.4}$$

Using (4.4) in place of (4.1), one can also show that for any $u \in C^\infty(\overline{\Omega})$:

(1) For any $k \in \mathbb{N}$ and smooth $(1,0)$ vector field Y satisfying $Y\rho = 0$ on $\partial\Omega$

$$\|Yu\|_{k-1}^2 \leq C_k \left(\left\|\bar{\partial} u\right\|_{k-1}^2 + \|u\|_{k-1} \|u\|_k \right). \tag{4.5}$$

(2) If u is compactly supported in Ω and $k \in \mathbb{N}$

$$\|u\|_k^2 \leq C_k \left(\left\|\bar{\partial} u\right\|_{k-1}^2 + \|u\|_{k-1}^2 \right). \tag{4.6}$$

Let v_ε, θ_ε and λ_ε be as in Proposition 3.4. Choose $\chi_\varepsilon \in C_0^\infty(\mathrm{supp}(\theta_\varepsilon))$ such that $\chi_\varepsilon \equiv 1$ on a neighborhood of $\partial\Omega$ contained in $\{z : \theta_\varepsilon(z) = \theta^{v_\varepsilon}(z)\}$ (one can use the same χ_ε from the proof of Proposition 3.4). We can estimate the Sobolev norm on $W^k_{(0,q)}(\Omega)$ using (4.1) and (4.2):

$$\begin{aligned}\|u\|_k^2 \leq{}& C_k \left(\sum_{j=1}^{n} \left\| \frac{\partial u}{\partial \bar{z}_j} \right\|_{k-1}^2 + \sum_{j=1}^{n} \left\| \frac{\partial u}{\partial z_j} - \chi_\varepsilon \frac{\partial \rho}{\partial z_j} \frac{v_\varepsilon u}{v_\varepsilon \rho} \right\|_{k-1}^2 \right) \\ &+ C \left(\sum_{j=1}^{n} \left\| \left(\chi_\varepsilon \frac{\partial \rho}{\partial z_j} (v_\varepsilon \rho)^{-1} v_\varepsilon \right)^k u \right\|^2 \right) \\ \leq{}& C_{\varepsilon,k} \left(\left\|\bar{\partial} u\right\|_{k-1}^2 + \left\|\bar{\partial}^* u\right\|_{k-1}^2 + \|u\|_{k-1}^2 + \|u\|_{k-1} \|u\|_k + \left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|^2 \right).\end{aligned} \tag{4.7}$$

Hence, to estimate $\|u\|_k^2$, it will suffice to estimate $\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|^2$ for some value of ε. If u is a function, we use (4.4) and (4.5) to obtain the similar

$$\|u\|_k^2 \leq C_{\varepsilon,k} \left(\left\|\bar{\partial} u\right\|_{k-1}^2 + \|u\|_{k-1}^2 + \|u\|_{k-1} \|u\|_k + \left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|^2 \right). \tag{4.8}$$

For any integer $k \geq 1$, we define pointwise norms

$$|u|_k = \sum_{j=0}^{k} \sum_{I \in \{1,\dots,2n\}^j} \left| \frac{\partial^j}{\partial x_I} u \right|,$$

and

$$\left|\bar{\nabla} u\right|_k = \sum_{I \in \{1,\dots,2n\}^k} \sum_{\ell=1}^{n} \left| \frac{\partial^k}{\partial x_I} \frac{\partial}{\partial \bar{z}_\ell} u \right|.$$

Since $\|\cdot\|$, $\|\cdot\|_{t\lambda_\varepsilon}$ and $\|\cdot\|_{t,\varepsilon}$ are all comparable, we can use these interchangeably, as long as we note that the comparison constant will always depend on ε.

Observe that $[\bar{\partial}, (\chi_\varepsilon v_\varepsilon)^k] = k[\bar{\partial}, \chi_\varepsilon v_\varepsilon](\chi_\varepsilon v_\varepsilon)^{k-1} + \cdots$ where the remaining terms are derivatives of order at most $k-1$. Note that

$$\begin{aligned}[\bar{\partial}, \chi_\varepsilon v_\varepsilon] &= \bar{\partial}\chi_\varepsilon \wedge v_\varepsilon + \chi_\varepsilon[\bar{\partial}, v_\varepsilon] \\ &= (\bar{\partial}\chi_\varepsilon + \theta_\varepsilon) \wedge v_\varepsilon + \chi_\varepsilon[\bar{\partial}, v_\varepsilon] - \theta_\varepsilon \wedge v_\varepsilon.\end{aligned}$$

By definition $d\rho(\chi_\varepsilon[\bar{\partial}, v_\varepsilon] - \theta_\varepsilon \wedge v_\varepsilon) = 0$ on $\partial\Omega$ so $\chi_\varepsilon[\bar{\partial}, v_\varepsilon] - \theta_\varepsilon \wedge v_\varepsilon$is made up of tangential $(1,0)$-derivatives. Since $\bar{\partial}\chi_\varepsilon$ is compactly supported, we can apply (4.2) and (4.3) to obtain

$$\begin{aligned}\left\|[\bar{\partial}, (\chi_\varepsilon v_\varepsilon)^k]u\right\|_{t,\varepsilon} &\leq k\left\|\theta_\varepsilon \wedge v_\varepsilon(\chi_\varepsilon v_\varepsilon)^{k-1}u\right\|_{t,\varepsilon} \\ &\quad + C_{\varepsilon,k}\left(\left\|\bar{\partial}u\right\|_{k-1} + \left\|\bar{\partial}^* u\right\|_{k-1} + \|u\|_{k-1} + \sqrt{\|u\|_{k-1}\|u\|_k}\right).\end{aligned} \tag{4.9}$$

As characterized by the statement of Proposition 3.4, the norm $\|\theta_\varepsilon \wedge \cdot\|_{t,\varepsilon}$ satisfies special estimates, so we have

$$\begin{aligned}\left\|[\bar{\partial}, (\chi_\varepsilon v_\varepsilon)^k]u\right\|_{t,\varepsilon} &\leq \varepsilon k C\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \\ &\quad + C_{\varepsilon,k}\left(\left\|\bar{\partial}u\right\|_{k-1} + \left\|\bar{\partial}^* u\right\|_{k-1} + \|u\|_{k-1} + \sqrt{\|u\|_{k-1}\|u\|_k}\right).\end{aligned}$$

Hence, assuming all norms are finite, we can use (3.6) to show

$$\begin{aligned}\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} &\leq \left\|(I-P)(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} + \left\|P(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \\ &\leq \left\|\bar{\partial}(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t,\varepsilon} + \left\|P(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \\ &\leq \left\|(\chi_\varepsilon v_\varepsilon)^k \bar{\partial}u\right\|_{t,\varepsilon} + \varepsilon k C\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} + \left\|P(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \\ &\quad + C_{\varepsilon,k}\left(\left\|\bar{\partial}u\right\|_{k-1} + \left\|\bar{\partial}^* u\right\|_{k-1} + \|u\|_{k-1} + \sqrt{\|u\|_{k-1}\|u\|_k}\right).\end{aligned}$$

We summarize with the following Lemma:

Lemma 4.2. *Let $u \in C^\infty_{(0,q)}(\bar{\Omega}) \cap \mathrm{Dom}(\bar{\partial}^*)$, $1 \leq q \leq n-1$. If $\varepsilon \ll \frac{1}{k}$, we then have*

$$\begin{aligned}\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} &\leq C\left(\left\|(\chi_\varepsilon v_\varepsilon)^k \bar{\partial}u\right\|_{t\lambda_\varepsilon} + \left\|P(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon}\right) \\ &\quad + C_{\varepsilon,k}\left(\left\|\bar{\partial}u\right\|_{k-1} + \left\|\bar{\partial}^* u\right\|_{k-1} + \|u\|_{k-1} + \sqrt{\|u\|_{k-1}\|u\|_k}\right).\end{aligned}$$

If $u \in C^\infty(\bar{\Omega}) \cap (\ker\bar{\partial})^\perp$ and $\varepsilon \ll \frac{1}{k}$,

$$\begin{aligned}\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} &\leq C\left(\left\|(\chi_\varepsilon v_\varepsilon)^k \bar{\partial}u\right\|_{t\lambda_\varepsilon} + \left\|P(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon}\right) \\ &\quad + C_{\varepsilon,k}\left(\left\|\bar{\partial}u\right\|_{k-1} + \|u\|_{k-1} + \sqrt{\|u\|_{k-1}\|u\|_k}\right).\end{aligned}$$

Remark 4.3. For the purposes of this paper, $\varepsilon \ll \frac{1}{k}$ means that there exists a sufficiently small constant $C > 0$ independent of ε and k such that the appropriate results holds for all $\varepsilon < \frac{C}{k}$.

So far we have only addressed commutators of v_ε and $\bar{\partial}$, but we will also need to commute with $\bar{\partial}^*$. Let ϑ denote the formal adjoint of $\bar{\partial}$. Borrowing a technique from [16] (see also [17]), we define for any $(1,0)$ vector field $L = \sum_{j=1}^n L^j \frac{\partial}{\partial z_j}$ the operator

$$D_L u = -\sum_{j=1}^n \left(L^j d\bar{z}_j \wedge \vartheta u + \vartheta(L^j d\bar{z}_j \wedge u) \right)$$

on $(0,q)$-forms when $q \geq 1$ and

$$D_L u = -\sum_{j=1}^n \left(\vartheta(L^j d\bar{z}_j \wedge u) \right)$$

on functions. We immediately have $[D_L, \vartheta]u = 0$ when $q \geq 1$. We can compute

$$\begin{aligned} D_L u &= \sum_{j=1}^n \frac{\partial}{\partial z_j}(L^j u) - \sum_{I \in \mathcal{I}} \sum_{j,k=1}^n \left(\frac{\partial}{\partial z_k} L^j \right) u_{kI} d\bar{z}_j \wedge d\bar{z}_I \\ &= Lu + \sum_{j=1}^n \left(\frac{\partial}{\partial z_j} L^j \right) u - \sum_{I \in \mathcal{I}} \sum_{j,k=1}^n \left(\frac{\partial}{\partial z_k} L^j \right) u_{kI} d\bar{z}_j \wedge d\bar{z}_I \end{aligned}$$

when $q \geq 1$ and

$$D_L u = Lu + \sum_{j=1}^n \left(\frac{\partial}{\partial z_j} L^j \right) u$$

when $q = 0$. Hence the principal part of D_L is L, and

$$\left| ((D_L)^k - L^k) u \right| \leq C \left| u \right|_{k-1}.$$

The other difficulty with v_ε is that it is non-tangential. This can be corrected by defining

$$v_\varepsilon^\sharp = \frac{d\rho(\bar{v}_\varepsilon)}{d\rho(v_\varepsilon)} v_\varepsilon.$$

so that $v_\varepsilon - \bar{v}_\varepsilon^\sharp$ is tangential on $\bar{\Omega}$. Observe that

$$\left| ((\chi_\varepsilon v_\varepsilon)^k - (\chi_\varepsilon v_\varepsilon^\sharp)^k) u \right| \leq \left| 1 - \left(\frac{d\rho(\bar{v}_\varepsilon)}{d\rho(v_\varepsilon)} \right)^k \right| \left| (\chi_\varepsilon v_\varepsilon)^k u \right|$$

and

$$\left| 1 - \left(\frac{d\rho(\bar{v}_\varepsilon)}{d\rho(v_\varepsilon)} \right)^k \right| = \left| 1 - e^{-2ik \arg(d\rho(v_\varepsilon))} \right| \leq 2k \left| \arg(d\rho(v_\varepsilon)) \right|,$$

so

$$\left| ((\chi_\varepsilon v_\varepsilon)^k - (\chi_\varepsilon v_\varepsilon^\sharp)^k) u \right| \leq 2k\varepsilon \left| (\chi_\varepsilon v_\varepsilon)^k u \right|.$$

Hence, if $\varepsilon \ll \frac{1}{k}$, we can estimate

$$\begin{aligned} \left| (\chi_\varepsilon v_\varepsilon)^k u \right| &\leq C \left| (\chi_\varepsilon v_\varepsilon^\sharp)^k u \right|, \\ \left| (\chi_\varepsilon v_\varepsilon^\sharp)^k u \right| &\leq C \left| (\chi_\varepsilon v_\varepsilon)^k u \right|. \end{aligned} \tag{4.10}$$

We wish to modify this to preserve $\mathrm{Dom}(\bar\partial^*)$. We know that for smooth u, $u\in\mathrm{Dom}(\bar\partial^*)$ if and only if $\sum_{j=1}^n \frac{\partial\rho}{\partial z_j}u_{jI}=0$ on $\partial\Omega$ for all $I\in\mathcal{I}_{q-1}$. Observe that if $u\in\mathrm{Dom}(\bar\partial^*)$, then for any $I\in\mathcal{I}_{q-1}$ on $\partial\Omega$ we have

$$\begin{aligned}\sum_{j=1}^n \frac{\partial\rho}{\partial z_j}(D_L u)_{jI} &= \sum_{j,k=1}^n\left(\frac{\partial\rho}{\partial z_k}\frac{\partial}{\partial z_j}\left(L^j u_{kI}\right)-\frac{\partial\rho}{\partial z_j}\left(\frac{\partial}{\partial z_k}L^j\right)u_{kI}\right)\\ &=\sum_{j,k=1}^n\left(\frac{\partial}{\partial z_j}\left(\frac{\partial\rho}{\partial z_k}L^j u_{kI}\right)-\left(\frac{\partial}{\partial z_k}\left(\frac{\partial\rho}{\partial z_j}L^j\right)\right)u_{kI}\right)\\ &=\sum_{k=1}^n\left(L\left(\frac{\partial\rho}{\partial z_k}u_{kI}\right)-\left(\frac{\partial}{\partial z_k}L\rho\right)u_{kI}\right).\end{aligned}$$

For $(0,1)$-vector fields $\bar L=\sum_{j=1}^n \bar L^j\frac{\partial}{\partial\bar z_j}$ and $(0,q)$-forms u we define

$$\begin{aligned}D_{\bar L}u &= \bar L u-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k,\ell=1}^n \frac{\partial\rho}{\partial\bar z_\ell}\left(\frac{\partial}{\partial z_j}\bar L^\ell\right)u_{jI}\left((L\rho)^{-1}L^k d\bar z_k\right)\wedge d\bar z_I\\ &=\bar L u-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^n \overline{\theta_j^L}u_{jI}\left(\frac{\bar L\rho}{L\rho}L^k d\bar z_k\right)\wedge d\bar z_I\end{aligned}$$

when $q>0$ and $D_{\bar L}u=\bar L u$ when $q=0$. When $q>0$, $u\in\mathrm{Dom}(\bar\partial^*)$ and $I\in\mathcal{I}_{q-1}$ then on $\partial\Omega$ we have

$$\begin{aligned}\sum_{j=1}^n\frac{\partial\rho}{\partial z_j}(D_{\bar L}u)_{jI} &= \sum_{k=1}^n\left(\frac{\partial\rho}{\partial z_k}\bar L u_{kI}-\sum_{j=1}^n\frac{\partial\rho}{\partial\bar z_j}\left(\frac{\partial}{\partial z_k}\bar L^j\right)u_{kI}\right)\\ &=\sum_{k=1}^n\left(\bar L\left(\frac{\partial\rho}{\partial z_k}u_{kI}\right)-\left(\frac{\partial}{\partial z_k}\bar L\rho\right)u_{kI}\right).\end{aligned}$$

Now we may define $T_\varepsilon=\chi_\varepsilon(v_\varepsilon^\sharp-\bar v_\varepsilon)$ and $D_T=D_{v^\sharp}-D_{\bar v}$ (to keep the notation uncluttered, we will write $D_{v^\sharp}=D_{\chi_\varepsilon v_\varepsilon^\sharp}$ and $D_{\bar v}=D_{\chi_\varepsilon\bar v_\varepsilon}$) and note that since T_ε is tangential, we have

$$\sum_{j=1}^n\frac{\partial\rho}{\partial z_j}(D_T u)_{jI}=\sum_{k=1}^n\left(T_\varepsilon\left(\frac{\partial\rho}{\partial z_k}u_{kI}\right)-\left(\frac{\partial}{\partial z_k}T_\varepsilon\rho\right)u_{kI}\right)=0.$$

Hence D_T preserves $\mathrm{Dom}(\bar\partial^*)$.

Since $[D_{v^\sharp},\vartheta]=0$, we have $[D_T,\bar\partial^*]=-[D_{\bar v},\vartheta]$ and

$$[\vartheta,D_{\bar v}]u=\chi_\varepsilon[\vartheta,\bar v_\varepsilon]u+\chi_\varepsilon\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^n\left(\bar\theta_\varepsilon\right)_j\frac{\bar v_\varepsilon\rho}{v_\varepsilon\rho}v_\varepsilon^k\frac{\partial}{\partial z_k}u_{jI}d\bar z_I+\cdots,$$

so

$$[D_T,\bar\partial^*]u=\chi_\varepsilon[\vartheta,\bar v_\varepsilon]u+\chi_\varepsilon\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n\left(\bar\theta_\varepsilon\right)_j v_\varepsilon^\sharp u_{jI}d\bar z_I+\cdots,$$

where the suppressed terms are all of order zero. Since $[\vartheta, \bar{v}_\varepsilon]$ is a $(0,1)$-derivative, we can estimate

(4.11)
$$\begin{aligned}\left|[(D_T)^k, \bar\partial^*]u\right| &\leq C_{\varepsilon,k}(\left|\bar\nabla u\right|_{k-1} + |u|_{k-1}) + k\sum_{I\in\mathcal{I}_{q-1}}\left|\chi_\varepsilon\sum_{j=1}^n \left(\bar\theta_\varepsilon\right)_j (D_T)^{k-1} v_\varepsilon^\sharp u_{jI}\right|\\ &\leq C_{\varepsilon,k}(\left|\bar\nabla u\right|_{k-1} + |u|_{k-1}) + k\sum_{I\in\mathcal{I}_{q-1}}\left|\sum_{j=1}^n \left(\bar\theta_\varepsilon\right)_j (D_T)^{k} u_{jI}\right|.\end{aligned}$$

Finally, we note that

$$\left|((D_T)^k - (D_{v^\sharp})^k)u\right| \leq C_{\varepsilon,k}(\left|\bar\nabla u\right|_{k-1} + |u|_{k-1}). \tag{4.12}$$

Assume $u \in C^\infty_{(0,q)}(\bar\Omega) \cap \operatorname{Dom}(\bar\partial^*)$, $1 \leq q \leq n$. We will needed to estimate

$$\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar\theta_\varepsilon)_j((D_T)^k u)_{jI} d\bar z_I\right\|_{t\lambda_\varepsilon}$$

In order to apply Proposition 3.4, we need to consider the Bergman Projection of $(D_T)^k u$ and its orthogonal complement separately. For the first, we note that the Bergman Projection also preserves $\operatorname{Dom}(\bar\partial^*)$ and apply (3.8) to obtain

$$\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar\theta_\varepsilon)_j(P(D_T)^k u)_{jI} d\bar z_I\right\|_{t\lambda_\varepsilon} \leq \varepsilon C\left\|\bar\partial^*(D_T)^k u\right\|_{t\lambda_\varepsilon}$$

Applying (4.11) and (4.1) gives us

$$\begin{aligned}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar\theta_\varepsilon)_j(P(D_T)^k u)_{jI} d\bar z_I\right\|_{t\lambda_\varepsilon} &\leq k\varepsilon C\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar\theta_\varepsilon)_j((D_T)^k u)_{jI} d\bar z_I\right\|_{t\lambda_\varepsilon}\\ &+ \varepsilon C\left\|(D_T)^k\bar\partial^* u\right\|_{t\lambda_\varepsilon} + C_{\varepsilon,k}(\left\|\bar\partial u\right\|_{k-1} + \left\|\bar\partial^* u\right\|_{k-1} + \|u\|_{k-1}).\end{aligned}$$

For the complement, we first use (4.12), (4.10) and (4.1) to obtain

$$\begin{aligned}&\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar\theta_\varepsilon)_j((I-P)(D_T)^k u)_{jI} d\bar z_I\right\|_{t\lambda_\varepsilon} \leq\\ &\qquad C\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar\theta_\varepsilon)_j((I-P)(\chi_\varepsilon v_\varepsilon)^k u)_{jI} d\bar z_I\right\|_{t\lambda_\varepsilon}\\ &\qquad + C_{\varepsilon,k}(\left\|\bar\partial u\right\|_{k-1} + \left\|\bar\partial^* u\right\|_{k-1} + \|u\|_{k-1}).\end{aligned}$$

Substituting into (3.8) yields

$$\begin{aligned}\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar\theta_\varepsilon)_j((I-P)(D_T)^k u)_{jI} d\bar z_I\right\|_{t\lambda_\varepsilon} &\leq \varepsilon C\left\|\bar\partial(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon}\\ &+ C_{\varepsilon,k}(\left\|\bar\partial u\right\|_{k-1} + \left\|\bar\partial^* u\right\|_{k-1} + \|u\|_{k-1}).\end{aligned}$$

Applying (4.9) with the $\|\cdot\|_{t,\varepsilon}$ norm replaced by the $\|\cdot\|_{t\lambda_\varepsilon}$ norm, we have

$$\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^{n}(\bar\theta_\varepsilon)_j((I-P)(D_T)^k u)_{jI}d\bar z_I\right\|_{t\lambda_\varepsilon}\leq k\varepsilon C\left(\left\|\theta_\varepsilon\wedge(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon}+\left\|(\chi_\varepsilon v_\varepsilon)^k\bar\partial u\right\|_{t\lambda_\varepsilon}\right) + C_{\varepsilon,k}(\left\|\bar\partial u\right\|_{k-1}+\left\|\bar\partial^* u\right\|_{k-1}+\|u\|_{k-1}+\sqrt{\|u\|_k\|u\|_{k-1}}).$$

Combining the estimates for the Bergman Projection and its complement gives us

$$\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^{n}(\bar\theta_\varepsilon)_j((D_T)^k u)_{jI}d\bar z_I\right\|_{t\lambda_\varepsilon}\leq k\varepsilon C\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^{n}(\bar\theta_\varepsilon)_j((D_T)^k u)_{jI}d\bar z_I\right\|_{t\lambda_\varepsilon}+\varepsilon C\left\|(D_T)^k\bar\partial^* u\right\|_{t\lambda_\varepsilon} + k\varepsilon C\left(\left\|\theta_\varepsilon\wedge(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon}+\left\|(\chi_\varepsilon v_\varepsilon)^k\bar\partial u\right\|_{t\lambda_\varepsilon}\right) + C_{\varepsilon,k}(\left\|\bar\partial u\right\|_{k-1}+\left\|\bar\partial^* u\right\|_{k-1}+\|u\|_{k-1}+\sqrt{\|u\|_k\|u\|_{k-1}}).$$

We can conclude

Lemma 4.4. *Let $u\in C^\infty_{(0,q)}(\bar\Omega)\cap\mathrm{Dom}(\bar\partial^*)$, $1\leq q\leq n$. If $\varepsilon\ll\frac{1}{k}$, we have*

$$\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^{n}(\bar\theta_\varepsilon)_j((D_T)^k u)_{jI}d\bar z_I\right\|_{t\lambda_\varepsilon}\leq C\left(\varepsilon\left\|(D_T)^k\bar\partial^* u\right\|_{t\lambda_\varepsilon}+kB_\varepsilon\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon}+k\varepsilon\left\|(\chi_\varepsilon v_\varepsilon)^k\bar\partial u\right\|_{t\lambda_\varepsilon}\right) + C_{\varepsilon,k}(\left\|\bar\partial u\right\|_{k-1}+\left\|\bar\partial^* u\right\|_{k-1}+\|u\|_{k-1}+\sqrt{\|u\|_k\|u\|_{k-1}}).$$

5. Global Regularity

We are now ready to prove our main estimate. We note that many methods for proving global regularity involve integration by parts with $\bar\partial$ and $\bar\partial^*$ so that only one of the commutators $[\bar\partial, v_\varepsilon]$ or $[\bar\partial^*, v_\varepsilon]$ needs to be estimated (e.g., [4] or [17]). Since our inner products involve a weight with no uniform bounds, integration by parts would introduce uncontrollable error terms. McNeal has introduced an argument for dealing with such weights in [23], which uses the complex hessian of the weight to control the error terms involving the gradient of the weight. This is closely related to the methods used in Section 3 of the present paper. However, McNeal's method leads to a compactness estimate for $\bar\partial^* N$ in an unweighted L^2-space, which is equivalent to the same estimate for $N\bar\partial$. We are unable to prove anything that strong, so we will first prove estimates that apply to $N\bar\partial$ in weighted spaces and use these to obtain estimates for $\bar\partial^* N$ in weighted spaces. This approach seems to require that we estimate both $[\bar\partial, v_\varepsilon]$ and $[\bar\partial^*, v_\varepsilon]$.

Theorem 5.1. *Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded pseudoconvex domain that possesses a family of transverse vector fields satisfying Definition 2.5. Then for every $k \geq 0$, the $\bar{\partial}$-Neumann operator satisfies the Sobolev space estimate*

$$\|Nf\|_k \leq C_k \|f\|_k$$

for all $f \in W^k_{(0,q)}(\Omega)$, $1 \leq q \leq n$.

Proof. Fix a smooth defining function ρ for Ω that has strictly pseudoconvex level sets near $\partial\Omega$. Let $\Omega_\delta = \{z \in \Omega : \rho(z) < -\delta\}$. Since Ω_δ is a bounded smooth strictly pseudoconvex domain, $N_\delta : W^s_{(0,q)}(\Omega_\delta) \to W^s_{(0,q)}(\Omega_\delta)$ for all $s \geq 0$ and $1 \leq q \leq n$ by the classic result of Kohn [19, 20]. As shown in [3], this is equivalent to exact regularity for the Bergman Projection P_δ. Since the derivatives of ρ are uniformly bounded on Ω, all constants C which depend on derivatives of ρ (e.g., those in Lemma 4.1) can be chosen to be uniform on Ω_δ. For every $\varepsilon > 0$, the conclusions of Lemma 2.6 will hold uniformly on Ω_δ whenever $\partial\Omega_\delta \subset U_\varepsilon$, which will hold for all sufficiently small δ. Hence, any constants C_ε which depend on the vector field v_ε and weight function λ_ε can also be chosen to be independent of δ. Let v_ε, λ_ε, θ_ε and χ_ε be as in Proposition 3.4.

Let $u \in C^\infty_{(0,q)}(\bar{\Omega}_\delta) \cap \mathrm{Dom}(\bar{\partial}^*_\delta)$, $1 \leq q \leq n-1$. Using Lemma 4.2 to estimate $(\chi_\varepsilon v_\varepsilon)^k u$, we must consider $P_\delta(\chi_\varepsilon v_\varepsilon)^k u$. Using (4.10), we have

$$\left\|P_\delta(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq C \left\|P_\delta(D_{v^\sharp})^k u\right\|_{t\lambda_\varepsilon} + C_{\varepsilon,k} \|u\|_{k-1}.$$

Using (4.12) and (4.1) we have

$$\begin{aligned}\left\|P_\delta(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq C &\left\|P_\delta(D_T)^k u\right\|_{t\lambda_\varepsilon}\\ &+ C_{\varepsilon,k}(\left\|\bar{\partial}u\right\|_{k-1} + \left\|\bar{\partial}^*_\delta u\right\|_{k-1} + \|u\|_{k-1}).\end{aligned}$$

Since D_T and P_δ preserve $\mathrm{Dom}(\bar{\partial}^*_\delta)$, we have $P_\delta(D_T)^k u \in \mathrm{Dom}(\bar{\partial}^*_\delta)$, so we can substitute into (3.9)

$$\begin{aligned}\left\|P_\delta(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq C &\left\|\bar{\partial}^*_\delta(D_T)^k u\right\|_{t\lambda_\varepsilon}\\ &+ C_{\varepsilon,k}(\left\|\bar{\partial}u\right\|_{k-1} + \left\|\bar{\partial}^*_\delta u\right\|_{k-1} + \|u\|_{k-1}).\end{aligned}$$

Applying (4.11) and (4.1) yields

$$\begin{aligned}\left\|P_\delta(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq kC &\left\|\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n (\bar{\theta}_\varepsilon)_j((D_T)^k u)_{jI} d\bar{z}_I\right\|_{t\lambda_\varepsilon}\\ &+ C_{\varepsilon,k}(\left\|\bar{\partial}u\right\|_{k-1} + \left\|\bar{\partial}^*_\delta u\right\|_{k-1} + \|u\|_{k-1}),\end{aligned}$$

and substituting into Lemma 4.4 we have

$$\begin{aligned}&\left\|P_\delta(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq \\ &\quad kC\left(\varepsilon\left\|(D_T)^k\bar{\partial}^* u\right\|_{t\lambda_\varepsilon} + kB_\varepsilon\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} + k\varepsilon\left\|(\chi_\varepsilon v_\varepsilon)^k\bar{\partial}u\right\|_{t\lambda_\varepsilon}\right)\\ &\qquad + C_{\varepsilon,k}(\left\|\bar{\partial}u\right\|_{k-1} + \left\|\bar{\partial}^* u\right\|_{k-1} + \|u\|_{k-1} + \sqrt{\|u\|_k \|u\|_{k-1}}).\end{aligned} \tag{5.1}$$

Returning to Lemma 4.2, we obtain

$$\begin{aligned}\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq C(1+k^2\varepsilon)\left\|(\chi_\varepsilon v_\varepsilon)^k \bar\partial u\right\|_{t\lambda_\varepsilon} \\ + kC\left(\varepsilon\left\|(D_T)^k\bar\partial^* u\right\|_{t\lambda_\varepsilon} + kB_\varepsilon\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon}\right) \\ + C_{\varepsilon,k}\left(\left\|\bar\partial u\right\|_{k-1} + \left\|\bar\partial^* u\right\|_{k-1} + \|u\|_{k-1} + \sqrt{\|u\|_{k-1}\|u\|_k}\right)\end{aligned}$$

For $B_\varepsilon \ll \frac{1}{k^2}$, we have

(5.2)

$$\begin{aligned}\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq C\left((1+k^2\varepsilon)\left\|(\chi_\varepsilon v_\varepsilon)^k\bar\partial u\right\|_{t\lambda_\varepsilon} + k\varepsilon\left\|(D_T)^k\bar\partial^* u\right\|_{t\lambda_\varepsilon}\right) \\ + C_{\varepsilon,k}\left(\left\|\bar\partial u\right\|_{k-1} + \left\|\bar\partial^* u\right\|_{k-1} + \|u\|_{k-1} + \sqrt{\|u\|_{k-1}\|u\|_k}\right).\end{aligned}$$

When $q = n$, $P_\delta(\chi_\varepsilon v_\varepsilon)^k u = (\chi_\varepsilon v_\varepsilon)^k u$, so (5.2) follows immediately from (5.1) without invoking Lemma 4.2. Combining this with (4.7) we conclude that for $u \in C^\infty_{(0,q)}(\bar\Omega_\delta) \cap \mathrm{Dom}(\bar\partial^*_\delta)$ we have

$$\|u\|_k \leq C_{\varepsilon,k}\left(\left\|\bar\partial u\right\|_k + \left\|\bar\partial^*_\delta u\right\|_k + \|u\|_{k-1} + \sqrt{\|u\|_{k-1}\|u\|_k}\right).$$

At this point, v_ε and λ_ε are no longer part of the estimate, so we may fix $\varepsilon > 0$ dependent on k and write C_k in place of $C_{\varepsilon,k}$. Since $\sqrt{\|u\|_{k-1}\|u\|_k} \leq \epsilon\|u\|_k + \frac{1}{4\epsilon}\|u\|_{k-1}$ for any $\epsilon > 0$, we can conclude

$$\|u\|_k \leq C_k\left(\left\|\bar\partial u\right\|_k + \left\|\bar\partial^*_\delta u\right\|_k + \|u\|_{k-1}\right).$$

Induction on k gives us

$$\|u\|_k \leq C_k\left(\left\|\bar\partial u\right\|_k + \left\|\bar\partial^*_\delta u\right\|_k\right). \tag{5.3}$$

For the $q = 0$ case, let $u \in C^\infty(\bar\Omega_\delta) \cap (\ker\bar\partial)^\perp$. We again consider $P_\delta(\chi_\varepsilon v_\varepsilon)^k u$. By (4.10), we have

$$\left\|P_\delta(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq C\left\|P_\delta(D_{v^\sharp})^k u\right\|_{t\lambda_\varepsilon} + C_{\varepsilon,k}\|u\|_{k-1}.$$

Using (4.12) and (4.4) we have

$$\left\|P_\delta(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq C\left\|P_\delta(D_T)^k u\right\|_{t\lambda_\varepsilon} + C_{\varepsilon,k}(\|u\|_{k-1} + \left\|\bar\partial u\right\|_{k-1}).$$

Since $u \in (\ker\bar\partial)^\perp$, we can use Kohn's Formula to show $u = u - P_\delta u = \bar\partial^*_\delta N_\delta\bar\partial u$. Note that $(D_T)^k$ preserves the domain of $\bar\partial^*_\delta$, so since P_δ vanishes on the range of $\bar\partial^*_\delta$, we have $P_\delta(D_T)^k u = P_\delta[(D_T)^k, \bar\partial^*_\delta]N_\delta\bar\partial u$. Using (4.11) and (4.1), we have

$$\begin{aligned}\left\|P_\delta(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq kC\left\|P_\delta\sum_{I\in\mathcal{I}_{q-1}}\sum_{j=1}^n(\bar\theta_\varepsilon)_j((D_T)^k N_\delta\bar\partial u)_{jI}\,d\bar z_I\right\|_{t\lambda_\varepsilon} \\ + C_{\varepsilon,k}(\|u\|_{k-1} + \left\|N_\delta\bar\partial u\right\|_{k-1} + \left\|\bar\partial u\right\|_{k-1}).\end{aligned}$$

We know that P_δ is bounded in the weighted norm by (3.7), so we can apply Lemma 4.4 to obtain

$$\begin{aligned}\left\|P_\delta(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq kC(\varepsilon\left\|(D_T)^k u\right\|_{t\lambda_\varepsilon} + kB_\varepsilon\left\|(\chi_\varepsilon v_\varepsilon)^k N_\delta\bar\partial u\right\|_{t\lambda_\varepsilon}) \\ + C_{\varepsilon,k}(\|u\|_{k-1} + \left\|N_\delta\bar\partial u\right\|_{k-1} + \left\|\bar\partial u\right\|_{k-1} + \sqrt{\left\|N_\delta\bar\partial u\right\|_k\left\|N_\delta\bar\partial u\right\|_{k-1}}).\end{aligned}$$

Since $N_\delta\bar\partial u$ is a $(0,1)$-form, we may apply (5.2) to $N_\delta\bar\partial u$, yielding

$$\left\|P_\delta(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq k\varepsilon C(1+k^2B_\varepsilon)\left\|(D_T)^k u\right\|_{t\lambda_\varepsilon} + C_{\varepsilon,k}(\|u\|_{k-1} + \left\|N_\delta\bar\partial u\right\|_{k-1} + \left\|\bar\partial u\right\|_{k-1} + \sqrt{\left\|N_\delta\bar\partial u\right\|_k \left\|N_\delta\bar\partial u\right\|_{k-1}}).$$

Now we invoke Lemma 4.2 to obtain

$$\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq C\left\|(\chi_\varepsilon v_\varepsilon)^k\bar\partial u\right\|_{t\lambda_\varepsilon} + k\varepsilon C(1+k^2B_\varepsilon)\left\|(D_T)^k u\right\|_{t\lambda_\varepsilon} + C_{\varepsilon,k}(\|u\|_{k-1} + \left\|N_\delta\bar\partial u\right\|_{k-1} + \left\|\bar\partial u\right\|_{k-1} + \sqrt{\|u\|_k\|u\|_{k-1}} + \sqrt{\left\|N_\delta\bar\partial u\right\|_k \left\|N_\delta\bar\partial u\right\|_{k-1}}).$$

Using (4.12), (4.10), (4.4), and continuing to assume that $\varepsilon \ll \frac{1}{k}$ and $B_\varepsilon \ll \frac{1}{k^2}$, we have

$$\left\|(\chi_\varepsilon v_\varepsilon)^k u\right\|_{t\lambda_\varepsilon} \leq C\left\|(\chi_\varepsilon v_\varepsilon)^k\bar\partial u\right\|_{t\lambda_\varepsilon} + C_{\varepsilon,k}(\|u\|_{k-1} + \left\|N_\delta\bar\partial u\right\|_{k-1} + \left\|\bar\partial u\right\|_{k-1} + \sqrt{\|u\|_k\|u\|_{k-1}} + \sqrt{\left\|N_\delta\bar\partial u\right\|_k \left\|N_\delta\bar\partial u\right\|_{k-1}}).$$

We use (4.8) to obtain

$$\|u\|_k \leq C_k(\left\|\bar\partial u\right\|_k + \|u\|_{k-1} + \left\|N_\delta\bar\partial u\right\|_{k-1} + \sqrt{\|u\|_k\|u\|_{k-1}} + \sqrt{\left\|N_\delta\bar\partial u\right\|_k \left\|N_\delta\bar\partial u\right\|_{k-1}}),$$

where we have once again fixed $\varepsilon > 0$ based on k and dropped the dependence on ε from C_k. Applying (5.3) to $N_\delta\bar\partial u$ (for both k and $k-1$) we have

$$\|u\|_k \leq C_k(\left\|\bar\partial u\right\|_k + \|u\|_{k-1} + \sqrt{\|u\|_k\|u\|_{k-1}}).$$

Proceeding as in the $1 \leq q \leq n$ case, we conclude

$$\|u\|_k \leq C_k\left\|\bar\partial u\right\|_k \tag{5.4}$$

for all $u \in C^\infty(\overline{\Omega}) \cap (\ker\bar\partial)^\perp$.

Since $\bar\partial^*_\delta N_\delta$ is exactly regular, we can now use (5.3) and (5.4) to estimate $u = \bar\partial^*_\delta N_\delta f$ for any $f \in W^k_{(0,q)}(\Omega) \cap \ker\bar\partial$ with $1 \leq q \leq n$. Using formula (2) in [3], we can use this to estimate the Bergman Projection $P_\delta u$ for all $u \in W^k_{(0,q)}(\Omega)$ with $0 \leq q \leq n-1$ (estimates for $q = n$ are trivial). By the main result of [3], we conclude

$$\|N_\delta f\|_k \leq C_k\|f\|_k$$

for all $f \in W^k_{(0,q)}(\Omega)$, $1 \leq q \leq n$. Since C_k is independent of δ, we can take weak limits and obtain our result. □

6. Bounded Plurisubharmonic Exhaustion Functions

In this section, we will show that the existence of certain bounded plurisubharmonic exhaustion functions suffice for global regularity. This will generalize a known result of Kohn [21]. While Kohn's main theorem is primarily concerned with quantitative estimates, it does imply a global regularity condition. Suppose that for every $0 < \eta < 1$ there exists a smooth defining function ρ_η such that $-(-\rho_\eta)^\eta$ is a plurisubharmonic function on Ω. For a fixed smooth defining function r, define h_η by $\rho_\eta = h_\eta r$. Kohn's main theorem implies that the $\bar\partial$-Neumann operator is globally regular if $(1-\eta)\sup_{\partial\Omega}\left(1+\frac{|\nabla h_\eta|}{h_\eta}\right)^3$ can be made arbitrarily small. This is easily seen to be equivalent to the condition that

$$\liminf_{\eta\to 1^-}\sqrt[3]{1-\eta}\sup_{\partial\Omega}|\nabla\log h_\eta| = 0.$$

One consequence of Theorem 6.2 below is that this condition can be relaxed to

$$\liminf_{\eta\to 1^-}\sqrt{1-\eta}\sup_{\partial\Omega}|\nabla\log h_\eta| = 0.$$

Before we state our main result, we will define a pseudometric to measure the relationship between two defining functions ρ and r. At any point $p\in\partial\Omega$, we define $N_p\subset T_p^{(1,0)}(\partial\Omega)$ to be the null-space of the Levi-form.

Definition 6.1. Let $\Omega\subset\mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain with smooth defining functions ρ and r. Let h be the smooth, non-vanishing function defined on $\bar\Omega$ by $\rho = hr$ (see Lemma 1.1.3 in [8]). For any $p\in\partial\Omega$, we define the quantity

$$\sigma_p(r,\rho) = \sup_{\{T\in N_p : |T|=1\}} |T(\log h)|$$

to measure the pointwise relationship between two defining functions. When N_p is empty, we set $\sigma_p(r,\rho)=0$. We also define

$$\sigma(r,\rho) = \sup_{p\in\partial\Omega}\sigma_p(r,\rho)$$

to measure the global relationship. Observe that σ_p is bounded by the gradient of $\log h$, and since $\log h$ is a C^1 function and Ω is bounded, σ is guaranteed to be finite.

Theorem 6.2. *Let $\Omega\subset\mathbb{C}^n$ be a smooth bounded pseudoconvex domain with a smooth defining function r. Suppose that for every $0<\eta<1$ there exists a smooth defining function ρ_η such that $-(-\rho_\eta)^\eta$ is plurisubharmonic and*

$$\liminf_{\eta\to 1^-}\sqrt{1-\eta}\,\sigma(\rho_\eta, r) = 0.$$

Then Ω admits a family of transverse vector fields satisfying Definition 2.5.

Remark 6.3. Suppose, for example, that Ω admits a defining function which is plurisubharmonic on the boundary. This is known to imply the existence of an approximately holomorphic family of transverse vector fields by a result of Boas and Straube [4], and a result of Fornaess and Herbig [14] shows that the hypothesis of 6.2 are also satisfied. In their construction, $\sigma(\rho_\eta, r)\approx(1-\eta)$, so the requirements of Theorem 6.2 are much weaker.

Remark 6.4. Observe that if r_1 and r_2 are two different defining functions for Ω, then $\sigma(\rho_\eta, r_1)\le\sigma(\rho_\eta, r_2)+\sigma(r_2, r_1)$, so

$$\liminf_{\eta\to 1^-}\sqrt{1-\eta}\,\sigma(\rho_\eta, r_1)\le\liminf_{\eta\to 1^-}\sqrt{1-\eta}\,\sigma(\rho_\eta, r_2)$$

and vice-versa. Hence, the conditions of Theorem 6.2 do not depend on the choice of a defining function r.

Proof. Fix $\varepsilon>0$. For any $p\in\partial\Omega$, we define γ_p to be the unique path containing p that is normal to the level curves of r in a neighborhood of the boundary. We need two constants to choose an appropriate value of η. The first of these is $d=\sup_\Omega|z|$.

To define the second, we note that since r is a C^3 defining function, for any $p\in\partial\Omega$ we can find a constant $k_p>0$ such that

$$\sum_{j,\ell=1}^{n}\left(\left(a^j\bar{a}^\ell\frac{\partial^2 r}{\partial z_j\partial\bar{z}_\ell}\right)(z)-\left(a^j\bar{a}^\ell\frac{\partial^2 r}{\partial z_j\partial\bar{z}_\ell}\right)(p)\right)\leq k_p\left((-r)\sum_{j=1}^{n}\left|a^j\right|^2\right)(z) \tag{6.1}$$

for any $z\in\gamma_p$ and smooth vector field a. Since the γ_p vary continuously, k_p is continuous function on $\partial\Omega$, so we can define our second constant by $k=\sup_{\partial\Omega}k_p$. We now choose $0<\eta<1$ satisfying $\varepsilon=\sqrt{\frac{1-\eta}{\eta}(1+4d^2k)}$ (i.e. $\eta=\frac{1+4d^2k}{\varepsilon^2+1+4d^2k}$).

Define φ_η by $-(-\rho_\eta)^\eta=-e^{-\varphi_\eta}(-r)^\eta$. Observe that, with h defined as in Definition 6.1, $\varphi_\eta=-\eta\log h$. For each $p\in\partial\Omega$, choose orthonormal coordinates z^p in a neighborhood of p so that $z^p(p)=0$, $\frac{\partial r}{\partial z_j}(p)=0$ if $j\neq n$, and the Levi-form is diagonalized with $\{z_1^p,\dots,z_m^p\}$ representing the null space of the Levi-form, $0\leq m\leq n-1$. With vector fields $L_j=\frac{\partial}{\partial z_j^p}-2\left|\partial r\right|^{-2}\frac{\partial r}{\partial\bar{z}_j^p}\frac{\partial}{\partial z_j^p}$ for $1\leq j\leq n-1$ and $L_n=2\left|\partial r\right|^{-2}\sum_{j=1}^n\frac{\partial r}{\partial\bar{z}_j^p}\frac{\partial}{\partial z_j^p}$ we can define

$$v_p=e^{\varphi_\eta}L_n+\sum_{j=m+1}^{n-1}a^jL_j+rbL_n$$

for values of a^j and b to be chosen later. We have $dr(v_p)=e^{\varphi_\eta}+rb$ and hence

$$\begin{aligned}\theta^{v_p}|_p&=e^{-\varphi_\eta}\sum_{j=1}^{n}\frac{\partial r}{\partial z_j^p}\bar\partial(v_p^j)\\&=e^{-\varphi_\eta}\bar\partial(e^{\varphi_\eta}+rb)-e^{-\varphi_\eta}\sum_{j=1}^{n}v_p^j\bar\partial\left(\frac{\partial r}{\partial z_j^p}\right)\\&=\bar\partial\varphi_\eta+e^{-\varphi_\eta}b\bar\partial r\\&\quad-e^{-\varphi_\eta}\sum_{j=m+1}^{n-1}a^j\left(\frac{\partial^2 r}{\partial z_j^p\partial\bar{z}_j^p}d\bar{z}_j^p+\frac{\partial^2 r}{\partial z_j^p\partial\bar{z}_n^p}d\bar{z}_n^p\right)-\bar\partial\left(\frac{\partial r}{\partial z_n^p}\right).\end{aligned}$$

We can now choose a^j so that $\theta_j^{v_p}|_p=0$ for all $m+1\leq j\leq n-1$ and then choose b so that $\theta_n^{v_p}|_p=0$. When $1\leq j\leq m$ we are left with $\theta_j^{v_p}|_p=\frac{\partial\varphi_\eta}{\partial\bar{z}_j^p}-\frac{\partial^2 r}{\partial\bar{z}_j^p\partial z_n^p}$, but then since $\varphi_\eta=-\eta\log h$ we have $\left|\theta_j^{v_p}|_p\right|\leq\frac{\eta}{\sqrt{2}}\sigma(r,\rho_\eta)+C\left\|r\right\|_{C^2}$. Hence, $\left|\theta^{v_p}\right||_p\leq\sqrt{m\eta}\sigma(r,\rho_\eta)+C\left\|r\right\|_{C^2}$, so there exists a neighborhood U_p of p on which $\left|\theta^{v_p}\right|\leq\sqrt{n-1}\sigma(r,\rho_\eta)+C\left\|r\right\|_{C^2}$. Choose a finite covering of $\partial\Omega$ by these U_p and a partition of unity χ_p subordinate to this cover, so that we can define

$$v_\varepsilon=\sum_p\chi_pv_p.$$

Since $dr(v_p)=e^{\varphi_\eta}$ on $\partial\Omega$ regardless of the value of p, we immediately obtain $dr(v_\varepsilon)=e^{\varphi_\eta}$ on $\partial\Omega$. Hence, on $\partial\Omega$, we can compute

$$\theta^{v_\varepsilon}=\sum_p\left(\chi_p\theta^{v_p}+\bar\partial\chi_p\right)=\sum_p(\chi_p\theta^{v_p}),$$

and hence

$$\left|\theta^{v_\varepsilon}\right|\leq\sqrt{n-1}\sigma(r,\rho_\eta)+C\left\|r\right\|_{C^2}.$$

By assumption (and the relationship between ε and η), we have on $\partial\Omega$

$$\lim_{\varepsilon\to 0^+} |\theta^{v_\varepsilon}|\,\varepsilon = \sqrt{1+4d^2k}\lim_{\eta\to 1^-}|\theta^{v_\varepsilon}|\sqrt{1-\eta} = 0.$$

Hence, there must exist constants $A_\varepsilon \to 0$ such that on $\partial\Omega$, $|\theta^{v_\varepsilon}| \le \frac{A_\varepsilon}{\varepsilon}$.

Apart from the bound on θ^{v_ε}, we will need one more fact about v_ε. Since $dr(v_\varepsilon) = e^{\varphi_\eta}$, we have on $\partial\Omega$ for any smooth $(0,1)$-vector field τ satisfying $dr(\tau)=0$ the following computation

(6.2)

$$\begin{aligned}\theta^{v_\varepsilon}(\tau) &= e^{-\varphi_\eta}(dr([\bar\partial, v_\varepsilon]))(\tau) = e^{-\varphi_\eta}\sum_{j=1}^n \frac{\partial r}{\partial z_j}\bar\partial v_\varepsilon^j(\tau)\\ &= -\sum_{j=1}^n v_\varepsilon^j\left(\bar\partial\left(\frac{\partial r}{\partial z_j}e^{-\varphi_\eta}\right)\right)(\tau) = \left(\bar\partial\varphi_\eta - e^{-\varphi_\eta}\sum_{j=1}^n v_\varepsilon^j\bar\partial\left(\frac{\partial r}{\partial z_j}\right)\right)(\tau).\end{aligned}$$

Now, we begin to construct λ_ε. First, we compute

$$\begin{aligned}i\partial\bar\partial(-(-\rho_\eta)^\eta) = {}& ie^{-\varphi_\eta}(-r)^\eta\partial\bar\partial\varphi_\eta - ie^{-\varphi_\eta}(-r)^\eta\partial\varphi_\eta\wedge\bar\partial\varphi_\eta\\ &- i\eta e^{-\varphi_\eta}(-r)^{\eta-1}(\partial r\wedge\bar\partial\varphi_\eta + \partial\varphi_\eta\wedge\bar\partial r)\\ &+ i\eta e^{-\varphi_\eta}(-r)^{\eta-1}\partial\bar\partial r + i\eta(1-\eta)e^{-\varphi_\eta}(-r)^{\eta-2}\partial r\wedge\bar\partial r.\end{aligned}$$

Since $i\partial\bar\partial(-(-\rho_\eta)^\eta)\ge 0$, we can divide this by $(-r)^\eta$ and $e^{-\varphi_\eta}$ and obtain

$$\begin{aligned}\text{(6.3)}\quad i\partial\bar\partial\varphi_\eta - i\partial\varphi_\eta\wedge\bar\partial\varphi_\eta - i\eta(-r)^{-1}(\partial r\wedge\bar\partial\varphi_\eta + \partial\varphi_\eta\wedge\bar\partial r)&\\ + i\eta(-r)^{-1}\partial\bar\partial r + i\eta(1-\eta)(-r)^{-2}\partial r\wedge\bar\partial r \ge 0.&\end{aligned}$$

Suppose that in a neighborhood of some point $p\in\partial\Omega$, τ_0 is a smooth $(1,0)$-vector field such that $\tau_0|_p$ is in N_p. By (6.1), τ_0 must satisfy

$$\sum_{j,\ell=1}^n\left(\left(\tau_0^j\bar\tau_0^\ell\frac{\partial^2 r}{\partial z_j\partial\bar z_\ell}\right)(z)\right) \le k\left((-r)\sum_{j=1}^n\left|\tau_0^j\right|^2\right)(z)$$

on γ_p. Applying (6.3) in the τ_0 direction at p (and remembering that $\partial r(\tau_0)|_p = 0$), we find that

$$\sum_{j,\ell=1}^n\tau_0^j\bar\tau_0^\ell\left(\frac{\partial^2\varphi_\eta}{\partial z_j\partial\bar z_\ell} - \frac{\partial\varphi_\eta}{\partial z_j}\frac{\partial\varphi_\eta}{\partial\bar z_\ell} + \eta k\delta_{j\ell}\right)\ge 0.$$

Now, if we restrict (6.3) to the span of τ_0 and v_ε, we obtain a positive 2×2 matrix. Computing the determinant of this matrix on $\gamma_p\backslash p$, we obtain

$$\begin{aligned}\left(\sum_{j,\ell=1}^n\tau_0^j\bar\tau_0^\ell\left(\frac{\partial^2\varphi_\eta}{\partial z_j\partial\bar z_\ell} - \frac{\partial\varphi_\eta}{\partial z_j}\frac{\partial\varphi_\eta}{\partial\bar z_\ell} + \eta k\delta_{j\ell}\right)\right)\eta(1-\eta)(-r)^{-2}\left|dr(v_\varepsilon)\right|^2 + O((-r)^{-1})&\\ \ge\left|\eta(-r)^{-1}\sum_{j,\ell=1}^n v_\varepsilon^j\bar\tau_0^\ell\left(\frac{\partial^2 r}{\partial z_j\partial\bar z_\ell} - \frac{\partial r}{\partial z_j}\frac{\partial\varphi_\eta}{\partial\bar z_\ell}\right)\right|^2 + O((-r)^{-1}).&\end{aligned}$$

If we multiply both sides by $(-r)^2$, we can restrict to the boundary and apply (6.2) to obtain

$$\left(\sum_{j,\ell=1}^{n}\tau_0^j\bar{\tau}_0^\ell\left(\frac{\partial^2\varphi_\eta}{\partial z_j\partial\bar{z}_\ell}-\frac{\partial\varphi_\eta}{\partial z_j}\frac{\partial\varphi_\eta}{\partial\bar{z}_\ell}+\eta k\delta_{j\ell}\right)\right)\eta(1-\eta)\left|dr(v_\varepsilon)\right|^2\geq\left|\eta e^{\varphi_\eta}\theta^{v_\varepsilon}(\bar{\tau}_0)\right|^2.$$

Canceling the $\left|dr(v_\varepsilon)\right|^2=e^{2\varphi_\eta}$ from both sides and dividing by $\eta(1-\eta)$ yields

$$\left(\sum_{j,\ell=1}^{n}\tau_0^j\bar{\tau}_0^\ell\left(\frac{\partial^2\varphi_\eta}{\partial z_j\partial\bar{z}_\ell}-\frac{\partial\varphi_\eta}{\partial z_j}\frac{\partial\varphi_\eta}{\partial\bar{z}_\ell}+\eta k\delta_{j\ell}\right)\right)\geq\frac{\eta}{1-\eta}\left|\theta^{v_\varepsilon}(\bar{\tau}_0)\right|^2 = \frac{1+4d^2k}{\varepsilon^2}\left|\theta^{v_\varepsilon}(\bar{\tau}_0)\right|^2. \tag{6.4}$$

To construct λ_ε, we define constants $A=\frac{1}{1+4d^2k}$ and $B=\frac{2k}{1+4d^2k}$ and let

$$\lambda_\varepsilon=A\varphi_\eta+B\left|z\right|^2+C_\eta r+D_\eta r^2,$$

with $C_\eta>0$ and $D_\eta>0$ to be chosen later. On $\partial\Omega$, we have

$$\begin{aligned}\partial\lambda_\varepsilon&=A\partial\varphi_\eta+B\partial\left|z\right|^2+C_\eta\partial r\\ i\partial\bar{\partial}\lambda_\varepsilon&=iA\partial\bar{\partial}\varphi_\eta+iB\partial\bar{\partial}\left|z\right|^2+iC_\eta\partial\bar{\partial}r+2iD_\eta\partial r\wedge\bar{\partial}r.\end{aligned}$$

We define

$$i\Theta=i\partial\bar{\partial}\lambda_\varepsilon-i\partial\lambda_\varepsilon\wedge\bar{\partial}\lambda_\varepsilon-i\frac{1}{\varepsilon^2}\bar{\theta}^{v_\varepsilon}\wedge\theta^{v_\varepsilon}$$

and claim that C_η and D_η can be chosen so that $i\Theta>0$ on $\partial\Omega$.

Fix $p\in\partial\Omega$. We assume the Levi-form of $\partial\Omega$ has both zero and strictly positive eigenvalues at p, since the following proof can be easily adapted for the other cases. Let τ_0 be a $(1,0)$-vector field in N_p. Using (6.4), we have

$$\begin{aligned}\sum_{j,\ell=1}^{n}\tau_0^j\bar{\tau}_0^\ell\Theta_{j\ell}&=\sum_{j,\ell=1}^{n}\tau_0^j\bar{\tau}_0^\ell\left(A\frac{\partial^2\varphi_\eta}{\partial z_j\partial\bar{z}_\ell}+B\delta_{j\ell}-\left(A\frac{\partial\varphi_\eta}{\partial z_j}+B\bar{z}_j\right)\left(A\frac{\partial\varphi_\eta}{\partial\bar{z}_\ell}+Bz_\ell\right)\right.\\&\qquad\left.-\frac{1}{\varepsilon^2}\bar{\theta}_j^{v_\varepsilon}\theta_\ell^{v_\varepsilon}\right)\\&\geq\sum_{j,\ell=1}^{n}\tau_0^j\bar{\tau}_0^\ell\left(A(1-A)\frac{\partial\varphi_\eta}{\partial z_j}\frac{\partial\varphi_\eta}{\partial\bar{z}_\ell}+(B-A\eta k)\delta_{j\ell}\right.\\&\qquad\left.-AB\left(\frac{\partial\varphi_\eta}{\partial z_j}z_\ell+\bar{z}_j\frac{\partial\varphi_\eta}{\partial\bar{z}_\ell}\right)-B^2\bar{z}_jz_\ell\right).\end{aligned}$$

Since $0<A<1$, we have

$$A(1-A)\frac{\partial\varphi_\eta}{\partial z_j}\frac{\partial\varphi_\eta}{\partial\bar{z}_\ell}-AB\left(\frac{\partial\varphi_\eta}{\partial z_j}z_\ell+\bar{z}_j\frac{\partial\varphi_\eta}{\partial\bar{z}_\ell}\right)\geq-\frac{AB^2}{1-A}\bar{z}_jz_\ell,$$

so we can substitute and obtain

$$\sum_{j,\ell=1}^{n}\tau_0^j\bar{\tau}_0^\ell\Theta_{j\ell}\geq\sum_{j,\ell=1}^{n}\tau_0^j\bar{\tau}_0^\ell\left((B-A\eta k)\delta_{j\ell}-\frac{AB^2}{1-A}\bar{z}_jz_\ell-B^2\bar{z}_jz_\ell\right).$$

Using $|z| \leq d$, this implies

$$\sum_{j,\ell=1}^{n} \tau_0^j \bar{\tau}_0^\ell \Theta_{j\ell} \geq \sum_{j,\ell=1}^{n} \tau_0^j \bar{\tau}_0^\ell \left((B - A\eta k)\delta_{j\ell} - \frac{B^2 d^2}{1-A} \delta_{j\ell} \right).$$

Substituting the known values for A and B, we can compute

$$(B - A\eta k) - \frac{B^2 d^2}{1-A} = \frac{k(1-\eta)}{1+4d^2 k}.$$

Hence, we conclude

$$\sum_{j,\ell=1}^{n} \tau_0^j \bar{\tau}_0^\ell \Theta_{j\ell} \geq \sum_{j=1}^{n} \frac{k(1-\eta)}{1+4d^2 k} \left| \tau_0^j \right|^2.$$

Next, we let τ_1 be a tangential $(1,0)$-vector field in the orthogonal complement of N_p. Since the Levi-form is positive definite at p, this necessarily implies

$$\sum_{j,\ell=1}^{n} \tau_1^j \bar{\tau}_1^\ell \frac{\partial^2 r}{\partial z_j \partial \bar{z}_\ell} > 0$$
$$\sum_{j,\ell=1}^{n} \tau_1^j \bar{\tau}_0^\ell \frac{\partial^2 r}{\partial z_j \partial \bar{z}_\ell} = 0.$$

In particular, this means that $\sum_{j,\ell} \tau_1^j \bar{\tau}_0^\ell \Theta_{j\ell}$ is independent of C_η and $\sum_{j,\ell} \tau_1^j \bar{\tau}_1^\ell \Theta_{j\ell}$ is an increasing linear function with respect to C_η, so we can choose C_η^p large enough to make $i\Theta$ strictly positive when restricted to the span of τ_1 and τ_0. Hence, $i\Theta$ can be made strictly positive on the span of all tangential $(1,0)$-vector fields at p. Since this is an open condition, it applies on an open neighborhood of p, so a standard compactness argument can be used to find a single value of C_η making $i\Theta$ strictly positive on all of $\partial\Omega$.

Similarly, we observe that for any smooth tangential $(1,0)$-vector field τ on $\partial\Omega$, $\sum_{j,\ell} \tau^j \bar{v}_\varepsilon^\ell \Theta_{j\ell}$ is independent of D_η and $\sum_{j,\ell} v_\varepsilon^j \bar{v}_\varepsilon^\ell \Theta_{j\ell}$ is an increasing linear function with respect to D_η, so D_η can be chosen large enough to make $i\Theta > 0$ on all of $\partial\Omega$. Since $i\partial\bar{\partial}\lambda_\varepsilon > i\partial\lambda_\varepsilon \wedge \bar{\partial}\lambda_\varepsilon$ is an open condition, it holds in a neighborhood of the boundary, and the proof of Lemma 3 in [25] can be used to extend λ_ε into the interior of Ω. □

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