FINITE UNIONS OF INTERPOLATING SEQUENCES FOR HARDY SPACES

DANIEL H. LUECKING

Abstract. A sequence which is a finite union of interpolating sequences for H^∞ have turned out to be especially important in the study of Bergman spaces. The Blaschke products $B(z)$ with such zero sequences have been shown to be exactly those such that the multiplication $f \mapsto fB$ defines an operator with closed range on the Bergman space. Similarly, they are exactly those Blaschke products that boundedly divide functions in the Bergman space which vanish on their zero sequence. There are several characterizations of these sequences, and here we add two more to those already known. We also provide a particularly simple new proof of one of the known characterizations. One of the new characterizations is that they are interpolating sequences for a more general interpolation problem.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk and $\mathbb{T}$ its boundary. Let m denote the normalized arc length measure on $\mathbb{T}$. For $0 < p \le \infty$ let L^p denote the usual Lebesgue space of all measurable functions f on $\mathbb{T}$ for which $|f|^p$ is integrable with respect to dm, and let H^p denote the usual Hardy space consisting of analytic functions f on $\mathbb{D}$ such that $\sup_{0<r<1} M_p(f,r) < \infty$, where

$$M_p(f,r) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} & 0 < p < \infty \\ \sup_{0\le\theta<2\pi} |f(re^{i\theta})| & p = \infty \end{cases}$$

We denote both the norm on L^p and the norm on H^p (defined by the supremum above) by $\|f\|_{H^p}$. It is well known that functions in H^p have radial limits $\lim_{r\to 1} f(re^{i\theta})$ for m-almost all θ and that this defines an isometry between H^p and a closed subspace of L^p. We abuse notation by writing f for both the function $f(z)$ on $\mathbb{D}$ and its limit $f(e^{i\theta})$ on $\mathbb{T}$. We also abuse the terminology by calliing $\|\cdot\|_{H^p}$ a norm even in the cases $0 < p < 1$.

Let $\psi(z,\zeta)$ denote the *pseudohyperbolic metric*:

$$\psi(z,\zeta) = \left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|.$$

We will use $D(z,r)$ for the *pseudohyperbolic disk* of radius r centered at z, that is, the ball of radius $r < 1$ in the pseudohyperbolic metric.

Let $\mathcal{Z} = \{z_k : k = 1,2,3,\dots\}$ be a sequence in $\mathbb{D}$ without limit points in $\mathbb{D}$. Define the space of sequences $l^p_{\mathcal{Z}}$, $0 < p \le \infty$, to be all those $w = (w_k)$ such that

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$\|w\|_{l^p_{\mathcal{Z}}} < \infty$, where

$$\|w\|^p_{l^p_{\mathcal{Z}}} = \begin{cases} \sum_j |w_k|^p(1-|z_k|^2) & \text{if } p < \infty \\ \sup_j |w_k| & \text{if } p = \infty \end{cases}$$

The usual H^p interpolation problem for $\mathcal{Z}$ is the following: given a sequence $w = (w_k) \in l^p_{\mathcal{Z}}$ find a function $f \in H^p$ such that $f(z_k) = w_k$ for all k. The sequence $\mathcal{Z}$ is called an *interpolating sequence for H^p* if every such interpolation problem has a solution. Interpolating sequences for H^∞ were characterized by L. Carleson in [1] and this was extended to all other H^p by H. S. Shapiro and A. Shields in [8]. Let T_p denote the interpolation operator: $T_p(f)$ is the sequence $(f(z_j) : j \geq 1)$. Note that the definition of interpolating sequence requires $l^p_{\mathcal{Z}} \subset T_p(H^p)$ but does not required $T_p(H^p) \subset l^p_{\mathcal{Z}}$, however in [8] it was shown that the former implies the latter.

The characterization of interpolating sequences is the same for all H^p: a sequence $\mathcal{Z}$ is interpolating for H^p $(0 < p \leq \infty)$ if and only if $\mathcal{Z}$ is *uniformly separated*, which means there exists $\delta > 0$ such that

$$\inf_k \prod_{j \neq k} \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right| > \delta.$$

This clearly implies that $\mathcal{Z}$ is *uniformly discrete*, which is to say $\psi(z_j, z_k) \geq \delta$ for all $j \neq k$. Closely associated with ψ are the *Moebius transformations*

$$\varphi_\zeta(z) = \frac{\zeta - z}{1 - \bar{\zeta} z}, \quad \zeta \neq 0.$$

Clearly $\psi(z, \zeta) = |\varphi_\zeta(z)|$.

Since an interpolating sequence for H^p is a zero sequence for some $f \in H^p$, it follows that an interpolating sequence $\mathcal{Z} = \{z_1, z_2, \dots\}$ is a *Blaschke sequence*, that is $\sum 1 - |z_k|^2 < \infty$. The *Blaschke product* $B_{\mathcal{Z}}$ determined by $\mathcal{Z}$ is

$$B_{\mathcal{Z}}(z) = z^m \prod_k \frac{\bar{z}_k}{|z_k|} \varphi_{z_k}(z)$$

where m is the number of repetitions of 0 in the sequence $\mathcal{Z}$. This product converges uniformly on compact sets in $\mathbb{D}$ if and only if $\mathcal{Z}$ is a Blaschke sequence. It provides a bounded analutic function that vanishes precisely on $\mathcal{Z}$, with the order of each zero equal to the number of repetitions of that point in the sequence $\mathcal{Z}$. When we say "$\mathcal{Z}$ is the zero sequence of f" we will always mean that the order of the zero of f at ζ equals the repetition of ζ in the sequence $\mathcal{Z}$. Also, "f vanishes on $\mathcal{Z}$" will mean f vanishes at ζ to order at least the repetition of ζ in $\mathcal{Z}$. Similar interpretations are to be given to statements such as "$\mathcal{Z} \subset \mathcal{W}$" when $\mathcal{Z}$ and $\mathcal{W}$ are sequences in $\mathbb{D}$.

The *Bergman space* A^p is defined to be the the collection of analytic functions f on $\mathbb{D}$ that belong to $L^p(dA)$ where dA denotes area measure on $\mathbb{D}$. The norm on A^p will be denoted $\|\cdot\|_{A^p}$. For $\alpha > -1$ let dA_α denote the measure defined by $dA_\alpha(z) = (1 + |z|^2)^\alpha \, dA(z)$. Then the *weighted Bergman space* $A^{p,\alpha}$ is the set of all analytic functions in $\mathbb{D}$ such that

$$\|f\|_{A^{p,\alpha}} \stackrel{\text{def}}{=} \left(\int_{\mathbb{D}} |f|^p \, dA_\alpha \right)^{1/p} < \infty.$$

In a recent paper, P. Duren and A Schuster [2] proved the following equivalence of five conditions. Some terms used here will be defined only in the next section. The unit point mass at a point a is denoted by δ_a.

Theorem A. *For a Blaschke sequence $\mathcal{Z} = \{z_k\}$ of points in $\mathbb{D}$, the following five statements are equivalent.*

(i) *$\mathcal{Z}$ is a finite union of uniformly separated sequences.*
(ii) *$\sum_{k=1}^{\infty}(1-|z_k|^2)\delta_{z_k}$ is a Carleson measure.*
(iii) *$\sup_{z\in\mathbb{D}}\sum_{k=1}^{\infty}(1-|\varphi_{z_k}(z)|^2) < \infty$.*
(iv) *The associated Blaschke product B is a universal divisor of A^p; that is, $f/B \in A^p$ for every function $f \in A^p$ that vanishes on $\mathcal{Z}$.*
(v) *The operator M_B of multiplication by the associated Blaschke product B is bounded below on A^p; that is, there is a constant $c > 0$ such that $\|Bf\|_{A^p} \geq c\|f\|_{A^p}$ for every function $f \in A^p$.*

In this paper we will provide a simpler proof of the fact that (iii) implies (iv). In addition, we prove the following sixth and seventh equivalent properties. The term *general interpolating sequence* will be defined later. Suffice it to say for now that it is a sequence for which a particular interpolation problem always has a solution in the indicated space of functions.

Theorem 1.1. *For a Blaschke sequence $\mathcal{Z} = \{z_k\}$ the following are equivalent.*

(i) *$\mathcal{Z}$ is a finite union of uniformly separated sequences.*
(vi) *For the associated Blaschke product B, the zero function does not belong to the closure of $\{B \circ \varphi_z \mid z \in \mathbb{D}\}$ in the topology of uniform convergence on compacta.*
(vii) *$\mathcal{Z}$ is a general interpolating sequence for H^p.*

In statements (iv), (v) and (vii) the value of p is ambiguous. As in Duren and Schuster's paper [2], it will follow from the proof that if any of the statements hold for one value of p then they all hold for all values of p.

We will prove that statement (v) of the Theorem A implies (vi) of Theorem 1.1 and that this implies (i). And then we will prove that (vii) of Theorem 1.1 is equivalent to the rest of the conditions.

2. Background

A measure $\mu \geq 0$ on $\mathbb{D}$ is called a *Carleson measure* if there is a finite constant $C \geq 0$ such that for every arc $I \subset \mathbb{T}$

$$\mu(S_I) \leq Cm(I)$$

where $S_I = \{z \in \mathbb{D} : z/|z| \in I \text{ and } 1-|z| < m(I)\}$ is the *Carleson square based on I*. The infimum of the set of C for which this inequality holds is denoted $\|\mu\|_*$. If any $0 < p < \infty$ is given, Carleson measures are precisely those for which there exists a constant C with

$$\int |f|^p \, d\mu \leq C\|f\|_{H^p}^p \quad \text{for all } f \in H^p.$$

In fact, there exist absolute constants a and B such that $a\|\mu\|_* \leq C \leq B\|\mu\|_*$.

Given the sequence $\mathcal{Z} = \{z_j : j \geq 1\}$ let $\mu_{\mathcal{Z}}$ denote the measure $\sum(1-|z_j|^2)\delta_{z_j}$. If $\mathcal{Z}$ is an interpolating sequence for H^p then, because it is a Blaschke sequence, $\mu_{\mathcal{Z}}$ is a finite measure. Another characterization of interpolating sequences (essentially also obtained in [1] and [8]) is that they must be uniformly discrete and $\mu_{\mathcal{Z}}$ must be a Carleson measure. A sequence where $\mu_{\mathcal{Z}}$ is a Carleson measure will be called a *Carleson sequence*. It is well know (see [7]) that they can be written as a finite union of uniformly discrete sequences, and so they are just the finite unions of interpolating sequences.

We can restate the definition of interpolating sequence in terms of the Banach spaces H^p and $l^p_{\mathcal{Z}}$. An interpolating sequences is one for which $l^p_{\mathcal{Z}}$ is contained in the range of the interpolation map $T_p : H^p \to l^p_{\mathcal{Z}}$. As already remarked, although it is not immediately obvious, once this range contains $l^p_{\mathcal{Z}}$, it is necessarily contained in it and the closed graph theorem implies T_p is necessarily continuous. Then, by the open mapping theorem, if $\mathcal{Z}$ is an interpolating sequence for H^p, there is a finite constant $M > 0$ such that every $w \in l^p_{\mathcal{Z}}$ is interpolated by some $f_w \in H^p$ with $\|f_w\|_\infty \le M\|w\|_\infty$. This fact can actually be obtained without the open mapping theorem, using a normal families argument. The smallest such constant M is called the *interpolation constant* of the sequence $\mathcal{Z}$.

An interpolating sequence necessarily has no repeated points and must be a Blaschke sequence. A Blaschke product whose zero sequence is an interpolating sequence is called an *interpolating Blaschke product.*

3. Finite Products of interpolating Blaschke

The earliest mention of finite products of interpolating Blaschke products that I am aware of is in the paper of McDonald and Sundberg [7], where they were part of the invertibility criterion for Toeplitz operators with analytic symbol on the Bergman space of $\mathbb{D}$.

In [2], P. Duren and A. Schuster compiled most of the known characterizations of these Blaschke products (equivalently, of Carleson sequences). They also provided a unified proof of these characterization in the sense that each condition was shown to imply the next in (almost) a cycle, and the proof of each implication was relatively painless.

In this section we discuss those conditions, plus one of the conditions in Theorem 1.1. Let $\mathcal{Z}$ be a Blaschke sequence (of distinct points) and let $B = B_{\mathcal{Z}}$ be the associated Blaschke product. For each $z_j \in \mathcal{Z}$, let $B_j(z)$ be the Blaschke product associated with $\mathcal{Z} \setminus \{z_j\}$. That is

$$B_j(z) = \prod_{k \neq j} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z}$$

Then a Blaschke sequence $\mathcal{Z}$ is uniformly separated if and only if

$$\epsilon = \inf_j |B_j(z_j)| > 0. \tag{3.1}$$

The following is easily seen to be equivalent to (3.1):

$$\epsilon = \inf_j (1 - |z_j|^2)|B'(z_j)| > 0.$$

This follows from the product rule for derivatives and the formula for the derivative of φ_{z_j}.

While the composition of a Blaschke product with a conformal selfmap of $\mathbb{D}$ is again a Blaschke product or a constant multiple of one, the Blaschke condition on the zero sequence is not uniform. If B is the Blaschke product with zero sequence $\mathcal{Z}$, and φ is a conformal selfmap, then $\varphi(\mathcal{Z})$ is the zero sequence of $B \circ \varphi^{-1}$. The

Blaschke condition of $\varphi(\mathcal{Z})$ is $\sum_j (1-|\varphi(z_j)|^2) < \infty$. We will say that $\mathcal{Z}$ satisfies a *uniform Blaschke condition* if

$$\sup_{\varphi} \sum_j (1-|\varphi(z_j)|^2) < \infty$$

where the supremum is taken over all conformal maps of $\mathbb{D}$ onto itself.

A Blaschke product B with zero sequence $\mathcal{Z}$ is called a *universal divisor* for A^p if there is a constant C such that $\|f/B\|_{A^p} \le C\|f\|_{A^p}$ whenever $f \in A^p$ vanishes on $\mathcal{Z}$ (counting multiplicity). By the closed graph theorem, the existence of the constant is implied by the seemingly weaker statement that $f/B \in A^p$ whenever $f \in A^p$ vanishes on $\mathcal{Z}$.

We define the operator M_B on A^p by $M_B f = Bf$. It clearly satisfies $\|M_B f\|_{A^p} \le \|f\|_{A^p}$ for any $0 < p < \infty$. We will say that M_B is *bounded below on A^p* if there exists a constant $c > 0$ such that $\|M_B f\|_{A^p} \ge c\|f\|_{A^p}$.

It is quite easy to create Blaschke products B such that for some sequence φ_{ζ_n} of conformal selfmaps of $\mathbb{D}$ one has $B \circ \varphi_{\zeta_n}$ tending to 0 uniformly on compact subsets of $\mathbb{D}$. For example, let $\{\zeta_n\}$ be a sequence of distinct points with $|\zeta_n| \to 1$ and satisfying $\sum n(1-|\zeta_n|) < \infty$. Let $\mathcal{Z}$ be the sequence containing each ζ_n repeated n times. Then if B is the associated Blaschke product, we see that $|B(\varphi_{\zeta_n}(z))|$ has a zero at 0 of order n, and so it has the form $z^n g(z)$ with $\|g\|_\infty = 1$. Therefore, it tends to 0 uniformly on compact. (This example also has the property the $\sum_{a\in\mathcal{Z}}(1-|\varphi_{\zeta_n}(a)|^2) \ge n$, so it idoes not satisfy the uniform Blaschke condition.)

We will say that B is *uniformly bounded away from zero* (or *uniformly nonzero* for short) if the above described phenomenon does not occur. That is, for any sequence $\varphi_n(z)$ of conformal selfmaps of $\mathbb{D}$ the compositions $B \circ \varphi_n$ do not tend to 0 uniformly on compacta.

Now we can state part of the content of Theorems A and 1.1 as the equivalence of the following conditions on $\mathcal{Z}$ and $B = B_{\mathcal{Z}}$:

(1) $\mathcal{Z}$ is a finite union of uniformly separated sequences.
(2) $\mathcal{Z}$ is a Carleson sequence.
(3) $\mathcal{Z}$ satisfies the uniform Blaschke condition.
(4) B is a universal divisor for A^p.
(5) M_B is bounded below on A^p
(6) B is uniformly nonzero.

In the next section we will provide a different proof that the uniform Blaschke condition on $\mathcal{Z}$ implies that $B_{\mathcal{Z}}$ is a universal divisor for A^p. In fact the proof works (and is no less simple) for weighted Bergman spaces $A^{p,\alpha}$. The proof that M_B is bounded below for any universal divisor is trivial in any space. The fact that B is uniformly nonzero when M_B is bounded below also has the same proof for any weighted Bergman space, and so the equivalence of all the conditions is not only independent of p but it is also true for any weighted $A^{p,\alpha}$ and is independent of α.

The last condition is probably the minimum conceivable necessary condition. All the other conditions (and the constants implicitly involved in them) are easily shown to be conformally invariant. Moreover, if we take each condition and fix the constant, it clearly implies that no limit of such Blaschke products can be zero. The proof will show that we get a slightly stronger result in one direction. The last condition can be replaced by the following: *For any subsequence ζ_n of $\mathcal{Z}$ the compositions $B \circ \varphi_{\zeta_n}$ do not tend to zero uniformly on compacta.* In other words, for sufficiency, we do not need to test all sequences of Moebius transformations, but only those of the form φ_ζ for $\zeta \in \mathcal{Z}$.

4. Proofs

We start with the simplified proof that (iii) implies (iv) in Theorem A. It is simpler in that it doesn't need any of the results of C. Horowitz' paper [4], using only basic inequalities.

Let f be a function in A^p vanishing on $\mathcal{Z}$ and suppose $\mathcal{Z}$ satisfies (iii). Let $\mathcal{W} = \{\zeta_1, \zeta_2, \dots\}$ be the zero sequence of f. Assume temporarily that $f(0) \neq 0$. We apply Jensen's formula writing it in the form:

$$\log|f(0)| + \sum_j \log\frac{r}{|\zeta_j|}\chi_{[|\zeta_j|,1)}(r) = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta.$$

If we sum only over the points in $\mathcal{Z}$ this equality becomes an inequality:

$$\log|f(0)| + \sum_j \log\frac{r}{|z_j|}\chi_{[|z_j|,1)}(r) \le \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta.$$

Now multiply this inequality by $2r\,dr$ and integrate with respect to r from 0 to 1:

$$\log|f(0)| + \sum_j\left[\log\frac{1}{|z_j|} - \frac{1-|z_j|^2}{2}\right] \le \frac{1}{\pi}\int_{\mathbb{D}}\log|f(w)|\,dA(w).$$

Now fix $\zeta \in \mathbb{D}$ with $f(\zeta) \neq 0$ and apply the above to $f \circ \varphi_\zeta$ where, as before, φ_ζ is the conformal map that interchanges 0 and z. The mapping φ_ζ is it's own inverse so the zeros of $f \circ \varphi_\zeta$ are $\varphi_\zeta(z_j)$:

$$\log|f(\zeta)| + \sum_j\left[\log\frac{1}{|\varphi_\zeta(z_j)|} - \frac{1-|\varphi_\zeta(z_j)|^2}{2}\right] \le \frac{1}{\pi}\int_{\mathbb{D}}\log|f(\varphi_\zeta(w))|\,dA(w).$$

Now we make a change of variable $w \mapsto \varphi_\zeta(w)$ in the integral on the right, and rearrange the parts of the inequality.

$$\log|f(\zeta)| + \log\frac{1}{|B(\zeta)|} \le \sum_j \frac{1-|\varphi_\zeta(z_j)|^2}{2} + \frac{1}{\pi}\int_{\mathbb{D}}\log|f(w)||\varphi_\zeta'(w)|^2\,dA(w).$$

Applying condition (iii), the sum on the right hand side is bounded by some constant C. Multiply this inequality by $p/2$, apply exp to both sides, and use the inequality of the geometric and arithmetic mean on the right side (the measure $|\varphi_\zeta'(w)|^2\,dA(w)/\pi$ has total measure 1):

$$\left|\frac{f(\zeta)}{B(\zeta)}\right|^{p/2} \le \frac{e^{Cp/2}}{\pi}\int_{\mathbb{D}}|f(w)|^{p/2}|\varphi_\zeta'(w)|^2\,dA(w).$$

Note that the right hand side can be seen as the application to an L^2 function (namely $|f|^{p/2}$) of an integral operator with a familiar kernel:

$$K(\zeta, w) = |\varphi_\zeta'(w)|^2 = \frac{(1-|\zeta|^2)^2}{|1-\bar{\zeta}w|^4}.$$

Condition (iv) will follow if we can show that this kernel $K(\zeta, w)$ defines a bounded integral operator on $L^2(dA)$. This was done in [5] using what is sometimes called the *Schur method*, sometimes the *Forelli-Rudin method*. It involves only Hölder's inequality and basic estimates of the integrals involved (see [3]). In [5] it was used to prove the boundedness of this integral operator in a wide variety of spaces. In particular, it is bounded in the weighted space $L^2(dA_\alpha)$. Thus condition (iv) of Theorem A (for any weighted Bergman space) follows from condition (iii).

Now let us prove part of Theorem 1.1 by showing that B is uniformly nonzero (condition (vi)) whenever M_B is bounded below on any A^p (condition v).

Let φ be a conformal self-map of $\mathbb{D}$ and consider the function $f(z) = \varphi'(z)^{2/p}$, which has norm $\|f\|_{A^p}^p = \int |\varphi'|^2\, dA = \int 1\, dA$ by a change of variables. Applying condition (v) we get

$$\int |B\circ\varphi|^p\, dA = \int |B|^p|\varphi'|^2\, dA = \int |Bf|^p\, dA = \|Bf\|_{A^p}^p \geq c^p\|f\|_{A^p} = c^p\pi.$$

If φ_n is a sequence of conformal maps, the above holds with φ equal to any one of them. Since $B\circ\varphi_n$ are uniformly bounded by 1, if it were to converge to 0 pointwise, the Lebesgue theorem would imply that their $L^p(dA)$ norms tend to 0, contradicting the above inequality. Thus B must be uniformly nonzero.

Finally, assuming $B_{\mathcal{Z}}$ is uniformly nonzero (condition (vi)) we wish to prove $\mathcal{Z}$ is a finite union of interpolating sequences (condition (i)).

First we show that condition (vi) implies we can factor B into a finite number of Blaschke products F_j such that the zero sequence $\mathcal{Z}_j$ of F_j satisfies $\psi(a_k, a_n) > 1/2$ for all $a_k, a_n \in \mathcal{Z}_j$, $k \neq n$.

Toward this end we show that there is an upper bound on the number of zeros of B contained in the pseudohyperbolic disks $D(\zeta, 1/2)$ as ζ varies in $\mathbb{D}$. Let $N(\zeta)$ denote the number of zeros a_k of B in $D(\zeta, 1/2)$. For any sequence ζ_n in $\mathbb{D}$, let φ_{ζ_n} be the conformal self-map that interchanges 0 and ζ_n and observe that $B\circ\varphi_{\zeta_n}$ has $N(\zeta_n)$ zeros in $|z| < 1/2$. If there is no upper bound on $N(\zeta)$, choose a sequence ζ_n with $N(\zeta_n) \to \infty$. Then we have $B\circ\varphi_{\zeta_n} \to 0$ uniformly on compacta. This contradicts (vi), so there must exist an N such that $N(\zeta) \leq N$ for all $\zeta \in \mathbb{D}$.

It is well-known that $\mathcal{Z}$ can now be partitioned into finitely many (in fact N) subsequences such that each subsequence has a separation of at least 1/2 (in the pseudohyperbolic metric) between points. Thus $\mathcal{Z} = \cup_1^N \mathcal{Z}_j$ and each $\mathcal{Z}_j$ satisfies $\psi(a, b) > 1/2$ for all pairs $a \neq b$ in $\mathcal{Z}_j$. This induces a factorization of $B = \prod_{j=1}^M F_j$. Let F be one of the factors and note that F must also be uniformly bounded away from zero. We want to show that the zero set of F is uniformly separated, so assume it is not. Then there is a sequence a_k of zeros of F such that $(1-|a_k|^2)|F'(a_k)| \to 0$. Without loss of generality we may assume the sequence $G_k = F\circ\varphi_{a_k}$ converges uniformly on compact sets to some function G. In the disk $|z| < 1/2$ each G_k has exactly one zero of order 1: at the origin. By Hurwitz' theorem, G must either have exactly one zero of order 1 (at the origin) or be identically zero. The latter is ruled out because F is uniformly nonzero. Therefore,

$$(1 - |a_k|^2)|F'(a_k)| = G_k'(0) \to G'(0) \neq 0,$$

and we have a contradiction.

5. General interpolation

In this section we set up the notions of interpolation we call *general* interpolation. This is nearly identical to the corresponding notion for Bergman spaces introduced in [6], except for the norm imposed on the sequence space. Therefore we will omit many of the (identical) proofs and most of the justification for certain choices.

For simplicity of notation, let $(S)_\epsilon$ denote the ϵ-neighborhood of a set S, in the pseudohyperbolic metric. That is $(S)_\epsilon = \bigcup_{s\in S} D(s,\epsilon)$.

Let $\mathcal{Z}$ be any sequence in $\mathbb{D}$ without limit points and let us suppose it has been partitioned into finite sets $\mathcal{Z}_k$ with the property that the pseudohyperbolic diameters of $\mathcal{Z}_k$ are bounded away from 1. Let there be given open sets G_k with the following properties: There is an upper bound $R < 1$ on the diameters of G_k and there is an $\epsilon > 0$ such that $(\mathcal{Z}_k)_\epsilon \subset G_k$. One could in fact take $G_k = (\mathcal{Z}_k)_\epsilon$, but we wish to start as general as possible.

Now we define a norm $\|\cdot\|_{G_k}$ for analytic functions on G_k to be the the supremum norm. That is, $\|f\|_{G_k} = \sup_{z\in G_k} |f(z)|$. This is probably the simplest choice of norms, but quite a lot of other choices produce the same meaning of general interpolation. Other norms that work (that is, ensure the validity of succeeding arguments) are L^p averages on G_k with respect to area or with respect to arc length on the boundary (assuming the pseudohyperbolic perimeter of the G_k are bounded). We stick with the notation $\|\cdot\|_{G_k}$ to emphasize that it need not always be the sup norm.

Let $\mathcal{A}_k$ be the space of all bounded analytic functions f on G_k with the sup norm and let $\mathcal{N}_k$ be the closed subspace of $\mathcal{A}_k$ of all functions that vanish on $\mathcal{Z}_k$ to order given by the repetition within $\mathcal{Z}_k$. Let $\mathcal{E}_k$ denote the quotient space $\mathcal{A}_k/\mathcal{N}_k$ with the quotient norm, denoted by $\|\cdot\|_k$.

Let d_k denote the Euclidean distance from G_k to the boundary of $\mathbb{D}$ and for $0 < p < \infty$ define X^p to be the space of all sequences (w_k) with $w_k \in \mathcal{E}_k$ that satisfy $\sum \|w_k\|_k^p d_k < \infty$. Let X^∞ denote the sequences (w_k) with $\sup \|w_k\|_k < \infty$. The notation X^p, for the sake of simplicity, obviously suppresses a great many details involved in the above description. The obvious norm in this space is denoted $\|\cdot\|_{X^p}$

We say a function $f \in H^p$ *interpolates* a sequence (w_k) if for each k, $f|_{G_k}$ lies in the equivalence class w_k. This is equivalent to saying that at points of $\mathcal{Z}$, f has the values (and values of derivatives, etc.) that are defined by w_k. Thus, this is truly interpolation. What makes it *general* is the more general sequence space X^p.

For example, if each $\mathcal{Z}_k = \{z_k\}$ is a singleton, the spaces $\mathcal{E}_k$ are one dimensional and therefore essentially $\mathbb{C}$. The equivalence class of $f \in \mathcal{A}_k$ is uniquely determined by the value of f at the point z_k. It is not hard to show that $\|f + \mathcal{N}_k\|_k$ is equivalent to $|f(z_k)|$, and so the above scheme is exactly the usual notion of interpolation (evaluating at points). The requirement that G_k have pseudohyperbolic diameters bounded away from 1 implies that d_k is equivalent to $1 - |a_k|$ for any choice of $a_k \in G_k$ so the space X^p is essentially $l^p_{\mathcal{Z}}$.

For another example, let n_k be a bounded sequence of nonnegative integers and let each $\mathcal{Z}_k$ be a single point z_k repeated n_k times. Then the equivalence class of $f \in \mathcal{A}_k$ is determined by the values of f, f',…,$f^{(n_k-1)}$ at z_k and we get the usual notion of *multiple interpolation* where we obtain our sequence by evaluating a function and its derivatives (up to some order) at z_k. It is not hard to show that the norm of the equivalence class $f + \mathcal{N}_k$ is essentially $|f(z_k)| + |f'(z_k)|d_k + \cdots + |f^{(n_k-1)}|d_k^{n_k-1}$.

We call a sequence $\mathcal{Z}$, with the above described partition and associated sequence space X^p, a *general interpolating sequence for H^p* if for each sequence $(w_k) \in X^p$ there is a function $f \in H^p$ that interpolates it (in the sense described above).

If we choose a different norm than the sup norm for $\|\cdot\|_{G_k}$ then the space X^p would not be the same (in general) and it would be conceivable that the collection of general interpolating sequences would change. However, it turns out that same collection results for any reasonable choice of norms. This is explored a little more fully in [6]. Here, suffice it to say that the facts outlined in the rest of this section remain valid if we changed, for example, to $L^p(dA)$ means over G_k or $L^p(ds)$ means over the boundary of $(\mathcal{Z}_k)_\epsilon$. This is not surprising, since these norms are dominated by the sup norm and also dominate the sup norm over slightly smaller sets.

We list a number of facts about general interpolating sequences. We omit the proofs where they are essentially the same as those in [6].

Fact 1. *General interpolating sequences for H^p are zero sequences for H^p.*

Fact 2. *If $\mathcal{Z} = \bigcup \mathcal{Z}_k$ is a general interpolating sequence for H^p then there exists $\delta > 0$ such that for all $j \neq k$, $\psi(\mathcal{Z}_j, \mathcal{Z}_k) > \delta$.*

That is, while we don't necessarily have uniform discreteness for $\mathcal{Z}$ and one can have $\psi(z_{k_i}, z_{j_i}) \to 0$, it can only happen if eventually z_{k_i} and z_{j_i} are in the same $\mathcal{Z}_{m_i}$.

Fact 3. *A subsequence of a general interpolating sequence is also a general interpolating sequence. In particular, selecting one point from each $\mathcal{Z}_k$ produces an ordinary interpolating sequence.*

A consequence of this (and the requirements on the diameters of G_k) is that there is an upper bound on the Carleson norms of all measures of the form $\sum(1-|a_k|)\delta_{a_k}$, where $a_k \in G_k$ is arbitrary. Therefore $\sum_k \|f|_{G_k}\|_\infty d_k \leq C\|f\|_{H^p}$ and so:

Fact 4. *For a general interpolating sequence, the interpolation operator T that takes $f \in H^p$ to the sequence (w_k) where w_k is the equivalence class of $f|_{G_k}$, is bounded onto X^p.*

If the interpolation operator is onto and bounded, then by the open mapping theorem there exists an interpolation constant M. That is, if $w \in X^p$ then there is a function $f_w \in H^p$ with norm at most $M\|w\|$ that interpolates it.

Fact 5. *If $\mathcal{Z} = \{z_1, z_2, \dots\}$ is a general interpolating sequence with constant M, $\eta > 0$, and z_0 is a point with $\psi(z_0, \mathcal{Z}) > \eta$, then $\mathcal{Z} \cup \{z_0\}$ is also a general interpolating sequence (selecting any disk $D(z_0, r)$ for the corresponding G_0) with interpolation constant depending only on M, p, η and r.*

Fact 6. *For any general interpolating sequence $\mathcal{Z}$, there is an upper bound N on the cardinality of every $\mathcal{Z}_k$.*

This last fact follows from the Moebius invariance of interpolating sequences. If it were not true we could take a $\mathcal{Z}_k$ with large cardinality, and use a Moebius transformation to move it near the origin. Then, using Fact 5 if necessary, we could obtain a function which is 1 at 0, zero at all points of $\mathcal{Z} \setminus \{0\}$ and with H^p norm less than the interpolation constant of $\mathcal{Z} \cup \{0\}$. This leads to a contradiction if the cardinalities are unbounded

6. The remaining proof

From Facts 3 and 6, it follows that a general interpolating sequence is a union of a finite number of ordinary interpolating sequences. Note that this necessary condition is independent of the choices involved: choosing how to divide $\mathcal{Z}$ into $\mathcal{Z}_k$ and choosing what open sets G_k are used.

What remains to be shown is the converse, that a Carleson sequence is a general interpolating sequence.

A Carleson sequence automatically has bounded density (that is, there is an upper bound on the number of points in any disk $D(z, R)$ for fixed R) and so, by [6], there exists a partition of $\mathcal{Z}$ into finite sets $\mathcal{Z}_k$ with these requirements satisfied: the sets $G_k = (\mathcal{Z}_k)_\epsilon$ are disjoint for some $\epsilon > 0$ and $\operatorname{diam}_\psi(G_k) \leq C$ for some C independent of k. Thus, the arguments of the rest of this section will not be vacuous.

The H^∞ case has essentially been know from the beginning. Any sequence $(w_k) \in X^\infty$ corresponds to a bounded analytic function on $G = \bigcup_k G_k$. Arc length on the boundary of G is easily seen to be a Carleson measure. It is then routine to show that (w_k) can be interpolated. The following merely fills in some of the details

Let $\mathcal{Z}_k$ and G_k be as described. Let w be an element of X^p and for each k let g_k be an analytic function on G_k in the equivalence class of w_k, such that $\|g_k\|_{G_k} = \|w_k\|_k$. Clearly there exists, for any J, an element p_J of H^∞ which interpolates w_k for $1 \leq k \leq J$ and which is in fact a polynomial. Using the usual duality methods, we can estimate the minimal H^∞ norm of such interpolators by

$$\sup \left| \int_{\mathbb{T}} h(z) p_J(z) \overline{B(z)}\, dz \right|$$

where B is the finite Blaschke product with zero sequence $\bigcup_{k=1}^J \mathcal{Z}_k$, and the supremum is taken over all functions h in the unit ball of H^1. Shrinking ϵ if necessary, we may assume the sets $(\mathcal{Z}_k)_\epsilon$ are disjoint (Fact 2). Let Γ_k be the boundary of $(\mathcal{Z}_k)_\epsilon$. By Cauchy's theorem, the above integral can be rewritten as the following integral over the set $\bigcup_{k=1}^J \Gamma_k$, made up of of piecewise smooth closed curves.

$$\sup \left| \int_{\mathbb{T}} h(z) p_J(z) \overline{B(z)}\, dz \right| = \sup \left| \sum_{k=1}^{J} \int_{\Gamma_k} \frac{h(z) p_J(z)}{B(z)}\, dz \right|$$

Since the integrals in the summand depend only on the equivalence classes of $(hp_J)|_{G_k}$ we get

$$\begin{aligned} \sup \left| \int_{\mathbb{T}} h(z) p_J(z) \overline{B(z)}\, dz \right| &\leq \sup \left| \sum_{k=1}^{J} \int_{\Gamma_k} \frac{h(z) g_k(z)}{B(z)}\, dz \right| \\ &\leq \sup_{1 \leq k \leq J} \frac{\|w_k\|}{\inf_{z \in \Gamma_k} |B(z)|} \int_{\bigcup \Gamma_k} |h(z)|\, dz. \end{aligned}$$

The infimum in the denominator above is bounded away from 0 independent of J for any ordinary interpolating sequence. Since $\mathcal{Z}$ is a finite union of ordinary interpolating sequences, that is true here as well. Let $\delta > 0$ be that lower bound. Moreover, it is clear that arc length on $\bigcup \Gamma_k$ is a Carleson measure with a constant bounded above independent of J. Let $C < \infty$ be that upper bound. Therefore, the minimal norm $f_J \in H^\infty$ that interpolates w_k for $1 \leq k \leq J$ satisfies

$$\|f_J\|_\infty \leq \frac{C}{\delta} \sup_k \|w_k\|.$$

Taking a limit point as $J \to \infty$ gives us an element of H^∞ that interpolates all w_k.

This proves the $p = \infty$ case in almost exactly the same manner as for ordinary interpolating sequences.

Now we turn to the case $0 < p < \infty$. For ordinary interpolating sequences one can construct an interpolation operator from a bounded sequence of bounded analytic functions F_k that satisfy $F_k(z_k) = 1$ and $F_k(z_j) = 0$ for $j \neq k$. This sequences is easily provided by the Blaschke product: $F_k(z) = c_k B_{\mathcal{Z}}/\varphi_{z_k}$ for appropriate constant c_k. In our case we need something similar: $F_k|_{G_k}$ is in an appropriate equivalence class, while $F_k|_{G_j}$ is equivalent to 0 for $j \neq k$. This is easily provided by the solution of the H^∞ case.

Let $w = (w_k) \in X^p$, so that $\|w\|_{X^p} = \sum \|w_k\|_k^p d_k < \infty$. For each k choose some $a_k \in G_k$, and consider the sum

$$f(z) = \sum_k F_k(z) \left(\frac{1 - |a_k|^2}{1 - \bar{a}_k z} \right)^2 e^{\beta_k(a_k) - \beta_k(z)} \tag{6.1}$$

where $F_k \in H^\infty$ will be chosen later and where

$$\beta_k(z) = \sum_{j \geq k} (1 - |a_j|^2) \frac{1 + \bar{a}_j z}{1 - \bar{a}_j z}.$$

The summand in (6.1), apart from the F_k, is essentially that of Vinogradov, Gorin, and Hruščëv in [9] where it was shown that there exists a constant C, depending only on the Carleson norm of the measure $\nu = \sum (1 - |a_k|)\delta_{a_k}$, such that

$$\sum_k \left| \frac{1 - |a_k|^2}{1 - \bar{a}_k z} \right|^2 e^{\mathrm{Re}(\beta_k(a_k) - \beta_k(z))} < C \tag{6.2}$$

It was also shown that $\mathrm{Re}\, \beta_k(a_k)$ is bounded, so the exponent $\mathrm{Re}(\beta_k(a_k) - \beta_k(z))$ in the above expression is bounded above uniformly in k. It is a consequence of Harnack's inequality that it is also bounded below when restricted to G_k, uniformly in k. It is clear that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |a_k|^2}{|1 - \bar{a}_k r e^{i\theta}|^2} \leq 1. \tag{6.3}$$

Setting $z = re^{i\theta}$ we view

$$\frac{1 - |a_k|^2}{(1 - \bar{a}_k r e^{i\theta})^2} e^{\beta_k(a_k) - \beta_k(re^{i\theta})}$$

as the kernel $G_r(\theta, \zeta)$ of an operator from $L^p(\nu)$ to $L^p(d\theta)$ (recall that $\nu = \sum (1 - |a_k|^2)\delta_k$). It satisfies (by (6.2) and (6.3))

$$\int |G_r(\theta, \zeta)| \, d\nu(\zeta) < C \quad \text{and} \quad \int |G_r(\theta, \zeta)| \, d\theta < C,$$

and so it defines a bounded operator between $L^p(d\nu)$ and $L^p(d\theta)$ for all $1 \leq p \leq \infty$. Thus we have

$$\int_0^{2\pi} \left(\sum_k b_k \left| \frac{1 - |a_k|^2}{1 - \bar{a}_k r e^{i\theta}} \right|^2 e^{\mathrm{Re}[\beta_k(a_k) - \beta_k(re^{i\theta})]} \, d\theta \right)^p \leq C \sum_k b_k^p d_k \tag{6.4}$$

for all sequences $b_k \geq 0$.

Give $(w_k) \in X^p$, we choose F_k so that $F_k|_{G_j}$ is equivalent to zero when $j \neq k$ and such that

$$F_k(z) \left(\frac{1 - |a_k|^2}{1 - \bar{a}_k z} \right)^2 e^{\beta_k(a_k) - \beta_k(z)}$$

is equivalent to a g_k in the class w_k. We can do this by choosing $F_k(z)$ equivalent to

$$g_k(z) \Bigg/ \left(\left(\frac{1-|a_k|^2}{1-\bar{a}_k z} \right)^2 e^{\beta_k(a_k)-\beta_k(z)} \right)$$

Since the denominator above has absolute value bounded below in G_k (independent of k) this can be done with $\|F_k\|_\infty < C\|w_k\|_k$ by the $p = \infty$ case. Now, since

$$\begin{aligned} |f(z)| &= \left| \sum_k F_k(z) \left(\frac{1-|a_k|^2}{1-\bar{a}_k z} \right)^2 e^{\beta_k(a_k)-\beta_k(z)} \right| \\ &\le C \sum_k \|w_k\|_k \left| \frac{1-|a_k|^2}{1-\bar{a}_k z} \right|^2 e^{\mathrm{Re}[\beta_k(a_k)-\beta_k(z)]} \end{aligned}$$

we have $\|f\|_{H^p} \le \|w\|_{X^p}$ by the inequality (6.4). Clearly also f interpolates w by the construction of F_k.

It remains to consider the case $p < 1$. In this case we perform the same construction with the sum

$$f(z) = \sum_k F_k(z) \left(\frac{1-|a_k|^2}{1-\bar{a}_k z} \right)^{2/p} e^{[\beta_k(a_k)-\beta_k(z)]/p}$$

Then estimating $\|f\|_{H^p}$ is a simple matter of taking all terms in the sum to the power p and integrating term by term. In all cases we obtain a function $f \in H^p$ that interpolates the given $w \in X^p$, and so $\mathcal{Z}$ is a general interpolating sequence. This completes the proof.

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Department of Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701

E-mail address: `luecking@uark.edu`