Fully non-linear PDE

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April 11, 2008
Fourth Lecture
Arkansas Spring Lecture Series

Fully non-linear equations on (M^n, g)

$$Rm = W \oplus \frac{1}{n-2} A \owedge g,$$

where

$$A = Ric - \frac{R}{2(n-1)} g$$

A: the Weyl-Schouten tensor.

Under $g_w = e^{2w} g$, $W_{g_w} = e^{-2w} W_g$.

$$A_{g_w} = A_g + (n-2)\{-\nabla^2 w + dw \otimes dw - \frac{|\nabla w|^2}{2} g\}.$$

Denote $\sigma_k(A_g)$ = k-th elementary function of eigenvalues of A_g.

Examples:

$$\begin{aligned}\sigma_1(A_g) &= \sum_i \lambda_i = \frac{n-2}{2(n-1)} R_g, \\ \sigma_2(A_g) &= \sum_{i<j} \lambda_i \lambda_j \\ &= \frac{1}{2}(|Tr\ A_g|^2 - |A_g|^2) \\ &= \frac{n}{8(n-1)} R^2 - \frac{1}{2}|Ric|^2, \\ \sigma_n(A_g) &= det(A_g)\end{aligned}$$

Equation of Monge-Ampere type: Study

$$\sigma_k(\nabla^2 u) = f > 0$$

Dirichlet problem for u defined on $\Omega \subset \mathbb{R}^n$
Caffarelli-Nirenberg-Spruck/ Kohn
Krylov, Evans
Pogorolev, Caffarelli
Garding
Trudinger-X. J. Wang

Fully Non-linear PDE: for $n = 4$, $k = 2$,

$$\sigma_2(\nabla^2 u) = \frac{1}{2}[(\Delta u)^2 - |\nabla^2 u|^2]$$

While for A_g:

$$\begin{aligned} \sigma_2(A_{g_w})e^{4w} = & \, \sigma_2(A_g) + 2[(\Delta w)^2 - |\nabla^2 w|^2 \\ & + (\nabla w, \nabla|\nabla w|^2) + \Delta w|\nabla w|^2)] \\ & + \text{lower order terms.} \end{aligned}$$

Geometric content of sign of $\sigma_k(A_g)$:

For $k = 1$, $\sigma_1(A_g) = c_n R_g$

Algebraic fact:

(i) When $n = 3$ and $\sigma_2(A_g) > 0$, then either $R_g > 0$ and the sectional curvature of g is positive or $R_g < 0$ and the sectional curvature of g is negative on M.

(ii) When $n = 4$ and $\sigma_2(A_g) > 0$, then either $R_g > 0$ and $Ric_g > 0$ on M or $R_g < 0$ and $Ric_g < 0$ on M.

Present Proof

(iii) For general n and $\sigma_i(A_g) > 0$ $\forall 1 \leq i \leq k$ (i.e. $A_g \in \Gamma_k^+$) for some $k \geq \frac{n}{2}$, then $Ric_g > 0$.

For $k = 2$ on (M^n, g), $n = 4$,

$$\sigma_2(A_g) = \frac{1}{6}R^2 - \frac{1}{2}|Ric|^2$$

$$Q_g = \frac{-1}{12}\Delta R_g + \frac{1}{2}\sigma_2(A_g).$$

Gauss-Bonnet-Chern formula

$$4\pi^2\chi(M) = \int_M (Q_g + \frac{1}{8}|W_g|^2)dv_g.$$

$$8\pi^2\chi(M) = \int_M (\sigma_2(A_g) + \frac{1}{4}|W_g|^2)dv_g.$$

$\int_M \sigma_2(A_g)dv_g$ is a conformal invariant.

Theorem: (Chang-Gursky-Yang, '01, Gursky-Viaclovski '03)
On (M^4, g), assume

(i) $Y(M^4, g) > 0$;
(ii) $\int \sigma_2(A_g) dv_g > 0$;

then $\exists\, w \in \mathcal{C}^\infty(M)$, with $\sigma_2(A_{g_w}) \equiv 1$.

Corollary: Under (i),(ii), on (M^4, g), $\exists\, g_w = e^{2w} g$ with $Ric_{g_w} > 0$; hence $\pi_1(M^4)$ is finite.

Proof of Theorem:
Part I: existence part: Under (i) and (ii) solve

$$\sigma_2(A_{g_w}) = f, \qquad \text{for some } f > 0$$

It turns out a good function f is $|W|^2$.

Part II: regularity part: Deform f to constant by method of continuity and degree theory.

Analytic tools in the regularity part:

Step 1: Apply blow up argument to get hold of C^0 bound of w. The goal is to aviod the sequence:

$$w_\lambda(x) = log \frac{2\lambda}{|\lambda|^2 + |x - x_0|^2}$$

which satisfies $\sigma_2(A_{g_{w_\lambda}}) = 1$ on (S^4, g_c).

Assume w achieves maximal point at $x = 0$, choose $w_a(y) = w(ay) + loga$, where we choose $a^2 e^{2w_a}(0) = 1$. Suppose $maxe^{2w_k} \to \infty$, choose sequence of w_{a_k}, then the limit of some subsequence tends to w_∞. which satisfies:

Liouville Theorem $\sigma_2(g_{w_\infty}) = 1$ on $\mathbb{R}^4$ with $R_{g_{w_\infty}} \geq 0$, then $w_\infty = w_\lambda$.

This will contradict with the assumption that

$$\int_M \sigma_2(A_g) dv_g < \int_{S^4} \sigma_2(A_{g_c}) dv_{g_c}$$

Step II: Lower order term in the expression of A_{g_w} helps us to gain C^1 estimates when $R_{g_w} \geq 0$.

Local Estimates

(Guan-Wang '02, Sophie Chen '06)

Assume $g_w \in \Gamma_2^+$; then

$$max_{B(\frac{1}{2})}(|\nabla w|^2 + |\nabla^2 w|) \leq C(1 + max_{B(1)} e^{2w})$$

ϵ-regularity Theorem Assume $g_w \in \Gamma_2^+$, then there exists some $\epsilon > 0$ so that

$$\int_{B(r)} e^{nw} dv_g \leq \epsilon,$$

then

$$max_{B(\frac{r}{2})} e^{2w} \leq C\frac{1}{r^2}.$$

Thus we have C^1, C^2 bound of w, hence $C^{2,alpha}$ and C^∞ bound of w.

For general $\sigma_k(A_g)$, deformation and regularity results for $g \in \Gamma_k^+$

$k \geq 3$, $k \leq \frac{n}{2}$ $\int \sigma_k(A_g) dv_g$ is variational over $[g]$ iff M is locally conformally flat. Viaclovsky, Branson-Gover

Shen-Trudinger-X.J. Wang
Y. Li and A. Li
Guan- Lin-G.Wang
Gursky-Viaclovsky

Theorem Ge-Lin-G Wang, Catino-Djadli
On (M^3, g), assume $R_g > 0$ and $\int \sigma_2(A_g) dv_g > 0$ then (M^3, g) is differmorphic to (S^3, g_c).

Theorem (Viaclovsky '00)
Consider the functional

$$F_k(g) := \int_{M^n} \sigma_k(g) dv_g,$$

(i): When $k = 1$ *or* 2, and $2k < n$, F_k is variational in the conformal class of metrics $g_w \in [g]$ with fixed volume one, i.e. the extremal metric for the functional in this class of metrics, when achieved, satisfies the equation

$$\sigma_k(g_w) = \text{constant}.$$

(ii) When $k \geq 3$ and $2k < n$, assertion in (i) only holds when the manifold M^n is locally conformally flat.

(iii) When $k = 2, n = 4$, $F_2(g)$ is conformally invariant; while for $k = \frac{n}{2}$ and $k \geq 3$, $F_k(g)$ is conformally invariant only when the manifold (M^n, g) is locally conformally flat.

Theorem (Chang-H.Fang) There exists curvature invariant $v^{(2k)}(g)$, so that: $v^{(2k)}(g)$ agrees with (mutiple of) $\sigma_k(g)$ when (M^n, g) is locally conformally flat; the functional

$$\mathcal{F}_k(g) = \int_M v^{(2k)}(g) dvol_g / (\int_M dvol_g)^{\frac{n-2k}{n}},$$

then $\mathcal{F}_k$ is variational within the conformal class; i.e., the critical metric in $[g]$ satisfies the equation

$$v^{(2k)} = \text{constant}.$$

For $n = 2k$, $F_{\frac{n}{2}}(g)$ is constant in the conformal class $[g]$.

Remark: The case $k = 2n$ is established earlier by Graham-Juhl '06, actually

$$v^{(n)} = Q_n + divD_n$$

, thus $\int v^{(n)}(g) dv_g = \int Q_n(g) dv_g$.

For $k = 1, 2$ cases, the new curvature invariant $v^{(2k)}$ agree, up to a scale, with the well-studied curvature polynomial $\sigma_k(g)$. Actually we have

$$v^{(2)}(g) = -\frac{1}{2}\sigma_1(g),$$

$$v^{(4)} = \frac{1}{4}\sigma_2(g).$$

but

$$v^{(6)}(g) = -\frac{1}{8}\left[\sigma_3(g) + \frac{1}{3(n-4)}(P_g)^{ij}(B_g)_{ij}\right],$$

where $(B_g)_{ij} := \nabla^k\nabla^l W_{likj} + \frac{1}{2}R^{kl}W_{likj}$ is the Bach tensor of the metric.

Fefferman-Graham '85 proved that for any given (M^n, g_0), there is an extension, g^+, which is "asymptotically Poincare Einstein" in a neighborhood of M^n, i.e. on $[0, \epsilon) \times M^n$ for some positive ϵ.

$$g_+ = \frac{1}{r^2}(dr^2 + h_{ij}(r,x)dx^i dx^j),$$

where $h(r, \cdot)$ is a metric defined for $M_c := r = c \subset X$.

$$dvol_h(r,x) = \sqrt{\det h_{ij}(r,x)}dx^1 \cdots dx^n.$$

Expand the quantity $\sqrt{\frac{\det h_{ij}(r,x)}{\det h_{ij}(0,x)}}$ in an expansion near $r = 0$ as

$$\sqrt{\frac{\det h_{ij}(r,x)}{\det h_{ij}(0,x)}} = \sum_{k=0}^{\infty} v^{(k)}(x, h_0) r^k,$$

where $v^{(k)}(x, h_0)$ is a curvature invariant of the metric $h_0 := h_{ij}(0, \cdot)$.

Some basic facts about $v^{(2k)}$:

(1) $v^{(k)}$ vanished for k odd and $2k < n$.

(2) $\int_M v^{(n)} dvol_{h_0}$ is conformally invariant over the conformal class of metrics of $[h_0]$. (Related to "renormalized volume").

(3) When h_0 is locally conformally flat, then (by K. Skenderis and S. Solodukhin)

$$v^{(2l)}(h_0) = (-2)^{-l}\sigma_l(h_0).$$

(4) It turns out that (Fefferman-Graham '07) that all $v^{(2k)}$ are "pure Ricci" curvature–i.e. curvatures involve Ricci curvatures and their covariant derivatives.

(5) It all turns out (Graham) that under conformal change of metric $g_w = e^{2w}g$, all $v^{(2k)}(g_w)$ involves only second derivative of w !!

Main observation in the proof of the Theorem:

Given $g_t = e^{2t\phi} g$ a variation of metrics on M in $[g]$, denote $V = \int_M dvol_g$. There exist functions $r_t = re^{w(t,\dot)}$ on a neighborhood of M in X such that

$$g_+ = \frac{1}{r_t^2}(dr_t^2 + h_t(r_t, \cdot)),$$

where $h_t(c, \cdot)$ is a metric defined on $M_{t,c} = \{r_t = c\} \subset X$. Furthermore, we have the following asymptotic expansion

$$h_t = g_t + r_t^2 g_t^{(2)} + \cdots .$$

For $p \in [0, \epsilon) \times M = X_{0,\epsilon}$, for each t, we can assign a local coordinate chart $p = (r_t, x_t) \in X_{t,\epsilon}$

$$\begin{aligned} dvol_{g^+}(p) &= r_t^{-n-1} dr_t dvol_{h_t}(r_t, x_t) \\ &= r_t^{-n-1} \sqrt{\frac{\det h_t(r_t, x_t)}{\det g_t(x_t)}} dr_t dvol_{g_t}(x_t), \end{aligned}$$

and

$$\sqrt{\frac{\det h_t(r_t, x_t)}{\det g_t(x_t)}} = \sum_k v^{(k)}(x_t, g_t) r_t^k.$$

Denote

$$D = \frac{d}{dt}|_{t=0}.$$

Then

$$\begin{aligned} 0 &= D(dvol_{g^+}(p)) \\ &= D(r_t^{-n-1}(\sum_k v^{(k)}(x_t, g_t) r_t^k) dr_t dvol_{g_t}(x_t)), \end{aligned}$$

we use Lebnitz Rule and apply estimates of w to get

$$\begin{aligned} 0 &= \sum_k r^{k-n-1}\{D[v^{(k)}(x_t, g_t)] \\ &+ k\phi(x) v^{(k)}(x, g)\} dr \wedge dvol_g(x) \\ &+ \sum_k r^{k-n-1} dr \mathcal{L}_{F_r}[v^{(k)} dvol_g(x)]. \end{aligned}$$

Integrate over M_r which is identified to M via the canonical diffeormorphism, we get

$$\begin{aligned} dr \ \ \sum \ \ & r^{k-n-1}\int_M (\{D[v^{(k)}(x_t, g_t)] \\ + \ \ & k\phi(x)v^{(k)}(x,g)\}dvol_g(x)) = 0. \end{aligned}$$

$$\begin{aligned} & D[\mathcal{F}_k(g_t)] \\ = \ \ & \frac{1}{V^{1-\frac{2k}{n}}}\int_M D[v^{(2k)}(x,g_t)] \ dvol_g(x) \\ + \ \ & n\int_M v^{(2k)}(x,g)\phi(x) \ dvol_g(x) \\ - \ \ & \frac{n-2k}{nV}\int_M n\phi \ dvol_g \int_M v^{(2k)}(x,g) \ dvol_g(x) \\ = \ \ & \frac{n-2k}{V^{1-\frac{2k}{n}}}\int_M [v^{(2k)} - \frac{\int_M v^{(2k)}(g,x)dvol_g(x)}{V}]\phi \ \ dvol_g. \end{aligned}$$

Thus when $n > 2k$, the critical metric g of the functional $\mathcal{F}_k$ satisfies

$$v^{(2k)}(g) = constant.$$

When $n = 2k$, the computation shows that the functional is invariant under the conformal deformation.