

Partial Differential Equations in Conformal Geometry

Sun-Yung Alice Chang
Princeton University

April 10-12, 2008
Arkansas Spring Lecture Series

On (M^n, g) , a Riemannian metric $g = \sum g_{ij} dx^i dx^j$ gives rise to measurement of angles and lengths between vectors.

A metric \hat{g} is **conformal** to g if $\hat{g} = \rho g$ for some positive function ρ . \hat{g} and g give the same angle measurements, we also denote $\rho = e^{2w}$ and $g_w = e^{2w} g$.

- **Conformal invariants** in this talk means: Quantities which is invariant under "conformal change of metrics".
- Using methods in PDE to study conformal invariants.

Model case: compact surface

On (M^2, g) **Gauss-Bonnet formula**:

$$2\pi\chi(M) = \int_M K_g dv_g,$$

where $K_g =$ Gaussian curvature of (M, g) ,
 $\chi(M)$ the Euler characteristic of M .

Uniformization Theorem:

Classify (orientable) (M^2, g)

When $\int_M K_g dv_g > 0$, we can find $w \in \mathcal{C}^\infty$ so
that $K_{g_w} \equiv 1$; thus (M^2, g) diffeomorphic (S^2, g_c) .

When $\int_M K_g dv_g = 0$, solve $K_{g_w} \equiv 0$; thus (M^2, g)
diffeomorphic $(\mathbb{R}^2/\Gamma, |dx|^2)$.

When $\int_M K_g dv_g < 0$, solve $K_{g_w} \equiv -1$; thus
 (M^2, g) diffeomorphic $(H^2/\Gamma, h_c)$.

On (M^2, g) denote $g_w = e^{2w}g$, then

One can solve

$$(*) : \quad -\Delta_g w + K_g = K_{g_w} e^{2w} \text{ on } M$$

An important property of Δ_g is the "covariant property":

$$\Delta_{g_w}(\phi) = e^{-2w} \Delta_g(\phi),$$

for all smooth functions ϕ .

On (M^n, g) , the **conformal Laplacian** L_g

$$L_g = -\Delta_g + c_n R_g$$

where $c_n = \frac{n-2}{4(n-1)}$, and R_g denotes the scalar curvature of the metric g .

Euler equation for Sobolev inequality on (M^n, g) :

Yamabe equation:

$$L_g u = c_n R_{\bar{g}} u^{\frac{n+2}{n-2}}.$$

where the conformal metric $\bar{g} = u^{\frac{4}{n-2}} g$ for some positive function u .

● **Yamabe problem:**

Given (M^n, g) , find positive function u so that $R_{\bar{g}}$ a constant.

(Yamabe, Trudinger, Aubin, Schoen).

Effort to generalize the role played by Gaussian curvature to other curvature invariants.

(A): To curvature invariants of higher order, in particular a n-th order one on manifolds of order n, called the Q-curvature, $\int Q_g dv_g$ is a conformal invariant.

Advantage: Under conformal change of metric, each such Q is prescribed by an n-order linear operator called the (generalized) **Paneitz**, or **GJMS operator**.

(B) To curvature polynomials, in particular to polynomials which are elementary symmetric function of the **Schouten tensor** P

$$P_{ij} = \frac{1}{n-2} \left(R_{ij} + \frac{1}{1(n-1)} R_g g_{ij} \right)$$

Advantage: One can use methods fully non-linear PDE of Monge-Type to study these polynomials under conformal changes. The geometric content of the quantity is also more clear.

Outline of talks:

(I): A brief survey of the analytic tools to study the 2nd order Scalar curvature equation and Gaussian curvature equations.

(II): Q-curvature : definitions; some analytic content

(III): Q-curvature : some geometric content, relation to the concept of renormalized volume in conformal compact Einstein manifolds.

(IV): $\sigma_k(P)$ equations, and generalized notions.

(V): A conformal compactness result of metrics on 3+1 conformal compact Einstein manifolds. (joint work with [Sophie Chen](#) and [Paul Yang](#))

§ Analytic aspects: A blow up sequence of functions

Sobolev Embedding Theorem:

For all $v \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$

$$(*) \quad \Lambda \left(\int_{\mathbb{R}^n} |v|^p dx \right)^{\frac{2}{p}} \leq \int_{\mathbb{R}^n} |\nabla v|^2 dx.$$

- We say that $W_0^{1,2}(\mathbb{R}^n)$ embeds into $L^p(\mathbb{R}^n)$.
- By a dilation of $v(x)$ to $v(\lambda x)$, we see p in $(*)$ is $p = \frac{2n}{n-2}$.

The **best constant** Λ and the **extremal functions** v for $(*)$: Assume $v(x) = v(|x|) = v(r)$,

$$\begin{cases} v'' + \frac{n-1}{r}v' + \Lambda v^{\frac{n+2}{n-2}} = 0, \\ v(0) = a, \quad v'(0) = 0. \end{cases}$$

One solution is

$$\begin{cases} v(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}} \\ \Lambda = \frac{n(n-2)}{4}\omega_n^{2/n}, \end{cases}$$

where ω_n is the surface area of the unit sphere S^n . We then observe that the inequality is invariant under:

$$v \rightarrow v_\epsilon(x) = \epsilon^{\frac{2-n}{2}} v\left(\frac{x - x_0}{\epsilon}\right),$$

where $\epsilon > 0$ and x_0 is any point in \mathbb{R}^n . In other words, we have

$$v_\epsilon(x) = \left(\frac{2\epsilon}{\epsilon^2 + |x - x_0|^2}\right)^{\frac{n-2}{2}}$$

are all extremals for the Sobolev embedding (*), we have the following remarkable theorem.

Theorem: (Bliss, Talenti, T. Aubin)

The best constant in the Sobolev inequality

$$(*) \quad \Lambda \left(\int_{\mathbb{R}^n} |v|^p dx \right)^{\frac{2}{p}} \leq \int_{\mathbb{R}^n} |\nabla v|^2 dx.$$

for $p = \frac{2n}{n-2}$ is $\Lambda = \frac{n(n-2)}{4} \omega_n^{2/n}$. It is **only** realized by the functions v_ϵ .

Properties of v_ϵ : (fix $x_0 = 0$, $\epsilon > 0$,)

$$v_\epsilon(x) = \left(\frac{2\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{2}}$$

(i) $v_\epsilon(0) = \left(\frac{2}{\epsilon} \right)^{\frac{n-2}{2}} \rightarrow \infty$ as $\epsilon \rightarrow 0$,

(ii) $v_\epsilon(x) \rightarrow 0$, for all $x \neq 0$, as $\epsilon \rightarrow 0$,

(iii) $\int_{\mathbb{R}^n} |v_\epsilon(x)|^{\frac{2n}{n-2}} dx = \int_{\mathbb{R}^n} |v_1(x)|^{\frac{2n}{n-2}} dx$,

(iv) $\int_{\mathbb{R}^n} |\nabla v_\epsilon(x)|^2 dx = \int_{\mathbb{R}^n} |\nabla v_1(x)|^2 dx$.

PICTURE

Thus v_ϵ is a sequence of functions

- bounded in $W^{1,2}(\mathbb{R}^n)$,
- The weak limit as $\epsilon \rightarrow 0$ is the zero function;

Hence it does not have a convergent subsequence in $L^{\frac{2n}{n-2}}$.

- The embedding of the Sobolev space $W^{1,2}(\mathbb{R}^n)$ into $L^{\frac{2n}{n-2}}$ is **not** compact. This **lack of compactness** due to the non-compact group of translations and dilations of \mathbb{R}^n is the heart of the problem.

The Euler Lagrange equation for the extremal function satisfies:

$$-\Delta v = \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} \text{ on } \mathbb{R}^n.$$

Thus functions v_ϵ above are solutions.

Theorem: (Caffarelli-Gidas-Spruck)

v_ϵ are the only positive solutions of above equation.

We conclude:

- All critical points of the Sobolev embedding are minimal points.
- The positive solutions are unique up to dilations and translations.

§ Blow up sequence on the unit sphere S^n
 Consider stereographic projection.

$$\pi : (S^n - \text{north pole}) \rightarrow \mathbb{R}^n$$

$$\xi \xrightarrow{\pi} x(\xi)$$

Sending the north pole on S^n to ∞ ;

$\xi = (\xi_1, \xi_2, \dots, \xi_{n+1})$ is a point in $S^n \subset \mathbb{R}^{n+1}$,
 $x = (x_1, x_2, \dots, x_n)$, then $\xi_i = \frac{2x_i}{1+|x|^2}$ for

$$1 \leq i \leq n; \xi_{n+1} = \frac{|x|^2 - 1}{|x|^2 + 1}.$$

PICTURE

Suppose u is a smooth function defined on S^n ,
 note that the Jacobian of π^{-1} as

$$J_{\pi^{-1}} = \left(\frac{2}{1 + |x|^2} \right) I$$

$$v(x) = u(\xi(x)) \left(\frac{2}{1 + |x|^2} \right)^{\frac{n-2}{2}},$$

Sobolev inequality on S^n :

$$\Lambda \left(\int_{S^n} |u(\xi)|^{\frac{2n}{n-2}} d\sigma(\xi) \right)^{\frac{n-2}{2}} \leq \int_{S^n} |\nabla u(\xi)|^2 d\sigma(\xi) + \frac{n(n-2)}{4} \int_{S^n} |u(\xi)|^2 d\sigma(\xi),$$

where $d\sigma(\xi) = \left(\frac{2}{1+|x|^2}\right)^n$ is the standard area form on the unit sphere S^n .

The transformed function $u(\xi)$ satisfies:

$$-\Delta_g u + \frac{n(n-2)}{4} u = \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} \text{ on } S^n,$$

where $\Delta_g = \left(\frac{2}{1+|x|^2}\right)^2 \Delta_x$.

- **Uniqueness** Functions u_ϵ obtained from v_ϵ are the only positive solutions.

On (M^n, g) , the **conformal Laplacian** L_g

$$L_g = -\Delta_g + c_n R_g$$

where $c_n = \frac{n-2}{4(n-1)}$, and R_g denotes the scalar curvature of the metric g .

Euler equation for Sobolev inequality on (M^n, g) :

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where the conformal metric $\bar{g} = u^{\frac{4}{n-2}} g$ for some positive function u .

● **Yamabe problem:**

Given (M^n, g) , find positive function u so that $R_{\bar{g}}$ a constant.

(Yamabe, Trudinger, Aubin, Schoen).

Variational method:

Find the extremals for the inequality:

$$\Lambda_g \left(\int_M |u|^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{2}} \leq \int_M |\nabla_g u|^2 dv_g + c_n \int_M R_g |u|^2 dv_g,$$

for some constant $\Lambda_g \leq \Lambda$.

- This constant Λ_g is called the **Yamabe constant**, and is conformally invariant.

A crucial ingredient in the proof: to establish some criteria for compactness of the minimizing sequence. That is to distinguish the manifold from the standard sphere by establishing $\Lambda_g < \Lambda_{g_c}$.

- In the solution by **Aubin**, the non-vanishing of the **Weyl tensor** in high dimensions plays this crucial role.

- Schoen uses the **positive mass theorem** to differentiate the conformal structure from the standard n -sphere.

Mass associated to a point p is defined as the finite part A in the asymptotic expansion of the Green's function of the conformal Laplacian with pole at p : in a geodesic coordinate system x whose origin is the given pole p , the Green's function G is the solution of the equation

$$L_g G = (n - 2)\omega_{n-1}\delta_p.$$

Near the point p there is an expansion:

$$G(x) = |x|^{2-n} + A + O(|x|).$$

- $A \geq 0$, $A = 0$ if and only if $(M^n, g) = (S^n, g_c)$.

§ Moser-Trudinger Inequality

Sobolev embedding Theorem:

$$W_0^{1,q}(D) \hookrightarrow L^q \text{ with } \frac{1}{p} = \frac{1}{q} - \frac{1}{n}.$$

When $q = 2$, $p = \frac{2n}{n-2}$ for $n \geq 3$.

When $q = 2$, $n = 2$ $0 < p < \infty$, but $p \neq \infty$.

Example: Take D to be the unit ball B in \mathbb{R}^2 ,
 $w(x) = \log \left| \log \left(e - 1 + \frac{1}{|x|} \right) \right|$.

Theorem: (Moser, Trudinger)

Suppose D is a smooth domain in \mathbb{R}^2 , then there is a constant C , for all functions $w \in W_0^{1,2}(D)$ with $\int_D |\nabla w(x)|^2 dx \leq 1$, we have

$$\int_D e^{\alpha w^2}(x) dx \leq C|D|,$$

for any $\alpha \leq 4\pi$, with 4π being the best constant.

- Existence of extremal functions for Moser's inequality. ([Carleson-Chang](#))

- Linearized form of the inequality is useful:

$$\log \frac{1}{|D|} \int_D e^{2w} dx \leq \frac{1}{4\pi} \int_D |\nabla w|^2 dx.$$

- ([W.Chen and C. Li](#))

Suppose w is in $C^2(\mathbb{R}^2)$, with $e^{2w} \in L^1(\mathbb{R}^2)$, and satisfies the equation

$$-\Delta w = e^{2w} \text{ on } \mathbb{R}^2.$$

Then

$$w(x) = \log \frac{2\epsilon}{\epsilon^2 + |x - x_0|^2}$$

for some $\epsilon > 0$ and some $x_0 \in \mathbb{R}^2$.

§ Gaussian curvature on compact surface

- Recall on (M^2, g) a compact surface, we have $\Delta = \Delta_g$ and the Gaussian curvature $K = K_g$.

- Under the conformal change $g_w = e^{2w}g$,

$$(1) \quad -\Delta_g w + K_g = K_w e^{2w} \text{ on } M$$

K_w denotes the Gaussian curvature of (M, g_w) .

- The **Gauss-Bonnet Theorem**:

$$2\pi \chi(M) = \int_M K_w dv_{g_w}$$

where $\chi(M)$ is the Euler characteristic of M .

- **Uniformization Theorem** to classify compact closed surfaces can be viewed as finding solutions with $K_w \equiv -1, 0, \text{ or } 1$ according to the sign of $\int K dv_g$.

$$(1) \quad -\Delta_g w + K_g = K_w e^{2w} \text{ on } M$$

Variational Functional:

$$J[w] = \int_M |\nabla w|^2 dv_g + 2 \int_M K_g w dv_g - \left(\int_M K_g dv_g \right) \log \frac{\int_M dv_{g_w}}{\int_M dv_g}.$$

Nirenberg problem: Which functions can be the Gaussian curvature function K_w , in particular on (S^2, g_c) .

- **Kazdan-Warner**

On $S^2 = \{(\xi_1, \xi_2, \xi_3) \mid \sum_{i=1}^3 \xi_i^2 = 1\}$, there is an **obstruction** for the problem:

$$\int_{S^2} \nabla K_w \cdot \nabla \xi e^{2w} dv_{g_c} = 0.$$

Theorem: (Moser)

Any positive C^2 even function f (i.e. $f(\xi) = f(-\xi)$ for all $\xi \in S^2$) can be a Gaussian curvature function on (S^2, g_c) .

Theorem: (Onofri; T.Aubin) $J[w] \geq 0$

and $J[w] = 0$ precisely for conformal factors w of the form $e^{2w}g = T^*g$ where T is a Mobius transformation of the 2-sphere.

Leray-Schauder degree theory for (1):

(Chang-Yang, Chang-Gursky-Yang)

(C.C. Chen and C.S. Lin)

Assume f is a Morse function satisfying the (non-degenerate condition) $\Delta f(\xi) \neq 0$ at the critical points ξ of f ,

$$\text{degree} = \sum_{\nabla f(q)=0, \Delta f(q)<0} (-1)^{\text{ind}(q)} - 1.$$

§ Geometric content of the functional $J[w]$

Polyakov-Ray-Singer Formula

On (M^2, g)

$$J[w] = 12\pi \log \left(\frac{\det(-\Delta_g)}{\det(-\Delta_{g_w})} \right)$$

where the determinant of the Laplacian $\det(-\Delta_g)$ is defined by Ray-Singer as :

$$\log \det(-\Delta_g) := -\zeta'(0).$$

Definition

On compact Riemannian manifold (M^n, g) , consider eigenvalue of $-\Delta_g$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$$

and the zeta function

$$\zeta(s) := \sum_{\lambda_k \neq 0} \lambda_k^{-s},$$

Formal differentiation leads to

$$\zeta'(s) = \sum_{\lambda_k \neq 0} -(\log \lambda_k) \lambda_k^{-s}, \text{ i.e.}$$

$$\zeta'(0) = - \sum_{\lambda_k \neq 0} \log \lambda_k = - \log \prod_{k=1}^{\infty} \lambda_k.$$

Apply Mellin transform for all $x > 0$,

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-xt} t^{s-1} dt.$$

We can rewrite $\zeta(s)$ in terms of the Gamma function:

$$\begin{aligned} \zeta(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} \sum_{j=1}^{\infty} e^{-\lambda_j t} t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} (Z(t) - 1) t^{s-1} dt, \end{aligned}$$

where $Z(t)$ denotes the **Heat kernel**. The existence of $\zeta'(0)$ can be justified via Weyl's asymptotic formula of the heat kernel.

- Onofri's inequality is equivalent to the statement $\det(-\Delta_{g_c})$ is maximal among all metrics g on S^2 .
- **Osgood-Phillips-Sarnak** independently derived Onofri's inequality and established the C^∞ compactness of **isospectral** metrics on compact surfaces.
- **Okikiolu**: Among all metrics with the same volume as the standard metric on the 3-sphere, the standard canonical metric is a **local** maximum for the functional $\det(-\Delta_g)$.

§4. Some Highlight of Lecture II: Study of Paneitz operator and Q-curvature on 4 manifold

On (M^n, g) , compact Riemannian manifold
A metric \bar{g} is conformal to g , if $\bar{g} = \rho g$ for some $\rho > 0$. Denote $\rho = e^{2w}$, and $g_w = e^{2w}g$.

An operator A is a **conformally covariant** operator of bidegree (a, b) , if under $g_w = e^{2w}g$,

$$A_{g_w}(\phi) = e^{-bw} A(e^{aw}\phi) \quad \forall \phi \in C^\infty(M^n).$$

Example:

When $n = 2$, Δ of bidegree $(0, 2)$. i.e. $\Delta_{g_w} = e^{-2w}\Delta_g$, and $-\Delta_g w + K_g = K_{g_w}e^w$,

Gauss-Bonnet formula:

$$2\pi\chi(M) = \int_M K_g dv_g$$

When $n = 4$, **Paneitz operator**: (1983)

$$P\varphi \equiv \Delta^2\varphi + \delta\left[\left(\frac{2}{3}Rg - 2Ric\right)d\varphi\right]$$

where δ denotes the divergence, d the deRham differential and Ric the Ricci tensor.

For example:

On $(R^4, |dx|^2)$, $P = \Delta^2$,

On (S^4, g_c) , $P = \Delta^2 - 2\Delta$,

On (M^4, g) , g Einstein, $P = (-\Delta) \circ (L)$.

P bidegree $(0, 4)$ on 4-manifolds, i.e.

$$P_{g_\omega}(\phi) = e^{-4\omega} P_g(\phi) \quad \forall \phi \in C^\infty(M^4).$$

Paneitz operator on (M^4, g) also satisfies

$$P_g w + 2Q_g = 2Q_{g_w} e^{4w}$$

$$Q = \frac{1}{12}(-\Delta R + R^2 - 3|Ric|^2)$$

Gauss-Bonnet-Chern Formula:

$$4\pi^2 \chi(M^4) = \int (Q_g + \frac{1}{8}|W_g|^2) dv,$$

where W denotes the Weyl tensor.

- $W_{g_w} = e^{-2w}W_g$ implies $|W_g|^2 dv_g = |W_{g_w}|^2 dv_{g_w}$
Thus $|W_g|^2 dv_g$ is a pointwise conformal invariant; and the curvature integral $\int Q_g dv_g$ is a conformal invariant.

Study geometric and analytic aspects of Q , relate to some fully non-linear PDE of Monge-Ampere type

Conformal Sphere Theorem:
(Chang-Gursky-Yang)

On (M^4, g) with $Y(M^4, g) > 0$.

(a) If

$$8\pi^2 \geq \int_{M^4} Q_g dv_g > 2\pi^2 \chi(M^4),$$

then M^4 is diffeomorphic to S^4 or $\mathbb{R}P^4$.

(b) If

$$\int_{M^4} Q_g dv_g = 2\pi^2 \chi(M^4),$$

and M^4 not diffeomorphic to S^4 or $\mathbb{R}P^4$ then either

- (1) (M^4, g) is conformal to $(\mathbb{C}P^2, g_{FS})$, or
- (2) (M^4, g) is conformal to $((S^3 \times S^1)/\Gamma, g_{prod})$.

§5 Some highlight of Lecture III:

- On (M^n, g) , existence results for P_{2k}^n , a differential operator of order $2k$, with leading term $(-\Delta_g)^k$ and conformal covariant of bi-degree $(\frac{n-2k}{2}, \frac{n+2k}{2})$.
- Construction method of Fefferman-Graham, Poincare metric g^+ on conformally compact Einstein manifold (X^{n+1}, M^n, g^+) .

Model case: (H^{n+1}, S^n, g_H) , the Poincare ball

- Work of [Graham-Zworski](#), relating P_{2k}^n and associated Q_{2k}^n -curvature to scattering matrix on (X^{n+1}, M^n, g^+)
- Discuss the concept of "renormalized volume" on (X^{n+1}, M^n, g^+) . When n is odd, relate the conformally invariant quantity to $\chi(X^{n+1})$, when n is even, relate the quantity to conformal primitive of Q_n^n .

Example: On $n = 2$, denote the renormalized volume by V , then

$$V_g - V_{g_w} = J[w] = 12\pi \log \left(\frac{\det(-\Delta_g)}{\det(-\Delta_{g_w})} \right),$$

where J is the Moser functional, which is the conformal primitive of the Gaussian curvature.