HMWK 3; DIFFERENTIAL GEOMETRY, FALL 2009

NAME :

Due on Monday, Oct. 19 2009
Justify all answers in detail!!!

Problem 1. Let A be a $n \times n$ matrix. Find an expression for

$$\frac{\partial \det A}{\partial A_{kl}}$$

Problem 2. Let A be a $n \times n$ matrix. Denote by $\mathrm{cof}A$ the matrix whose (ij) entry is $(-1)^{i+j}$ times the determinant of the submatrix obtained from A by erasing the $i-th$ row and the $j-th$ column. Find an expression for

$$\frac{\partial \mathrm{cof}A_{ij}}{\partial A_{kl}}$$

Hint: Use the fact $\mathrm{cof}A^T A = \det A I_n$.

Problem 3. Let A be a non-singular $n \times n$ matrix. Denote by A^{ij} the (ij) entry of A^{-1}. Find an expression for

$$\frac{\partial A^{ij}}{\partial A_{kl}}$$

Hint: Use the fact that $\det AA^{-1} = \mathrm{cof}A^T$.

Problem 4. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 map. Find an expression for

$$\frac{\partial (\mathrm{cof}DF)_{ij}}{\partial x_k}.$$

Show that

$$\sum_{j=1}^{n} \frac{\partial (\mathrm{cof}DF)_{ij}}{\partial x_j} = 0.$$

Problem 5. Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be two smooth maps. Assume that there exists $\epsilon > 0$ such that $\det DF(x) > \epsilon > 0$ for all $x \in B(0,1)$. Show that $DF(x) + sDG(x)$ is also non singular in $B(0,1)$ for $s \in \mathbb{R}$ in a sufficiently small neighborhood of zero.

Problem 5. Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be two smooth maps. Assume that there exists $\epsilon > 0$ such that $\det DF(x) > \epsilon > 0$ for all $x \in B(0,1)$. Use the chain rule to find an expression for

$$\frac{d}{ds} \frac{|D(F + sG)|^n}{\det(DF + sDG)}$$

in $B(0,1)$ and at $s = 0$.

Problem 6. Let $U \subset \mathbb{R}^n$ be an open set. A smooth, mapping $F : U \to \mathbb{R}^n$ is called *conformal* if there exists a scalar function $\lambda : U \to \mathbb{R}$ such that for all $x \in U$

$$DF^T DF = \lambda^2(x) I_n.$$

Show that the maps
- Rotations $x \to Rx$ with R a $n \times n$ matrix such that $R^T R = I_n$.
- Translations $x \to x + a$, for a fixed $a \in \mathbb{R}^n$.
- Inversion $x \to x/|x|^2$, for $x \neq 0$.

are all conformal.

Problem 7. Use the chain rule to show that the composition of the maps described above is still conformal. Can you find any other example of conformal maps?