Sampling and Oversampling in Shift-Invariant and Multiresolution Spaces I: Validation of Sampling Schemes

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Abstract

We investigate applications of ideas surrounding the Poisson summation formula and the Zak transform to the sampling of signals in principal shift-invariant spaces. The main question we ask is: what conditions need to be placed on the generators of these spaces to ensure that appropriate analogues of the classical sampling theorem for bandlimited signals apply to this new class of signals. The answer is shown to depend on general properties such as orthogonality, smoothness, self-similarity, bandlimitedess and compact support of the generator. Effective sampling rates which depend on the length of support of the generator or its Fourier transform are derived.

1 Introduction

The sampling problem asks: given a-priori conditions on an analog signal f and on a discrete set Λ, can all values of f be recovered once one knows the values on Λ? This is a classical problem in analysis. We impose on f the condition that it lie in the closed subspace $V(\varphi)$ of $L^2(\mathbb{R})$ generated by the shifts $\varphi(\cdot - k)$ ($k \in \mathbb{Z}$) of a fixed *principal* function φ. We then seek conditions on φ and on a set of sampling points in $\mathbb{R}$ such that the sampling problem has an affirmative answer. The condition that $f \in V(\varphi)$ generalizes the classical bandlimited assumption. Its relevance to specific signal processing applications will be discussed in the companion paper [9]. In any case, some a-priori signal constraint must be imposed to validate a sampling formula and such a constraint should be aligned with a signal model underlying applications one has in mind. That the sampling problem is ill-posed if one assumes nothing more than $f \in L^2(\mathbb{R})$ follows simply from the fact that f then is only defined up to a set of measure zero. At an absolute minimum, we require a model space X and lattice Λ such that the sampling operator $S_\Lambda : f \mapsto f|_\Lambda$ is well-defined. With such a space X and lattice Λ it is natural to ask:

(1) *Uniqueness:* Do the samples $f|_\Lambda$ determine f within X?

If one can answer this question affirmatively then it makes sense to proceed a step further and ask:

(2) *Reconstruction:* Can one reconstruct f in a stable way from its samples $f|_\Lambda$? More precisely, does there exist a family of functions $\varphi_\lambda \in X$ such that

$$f = \sum_{\lambda \in \Lambda} f(\lambda)\varphi_\lambda?$$

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A set Λ satisfying (1) is called a set of uniqueness for X, while if Λ satisfies (2), it is said to be a (stable) set of sampling for X. The uniqueness question already poses highly nontrivial constraints on the kind of function f can be. Stability is usually posed in terms of continuity of the reconstruction in some function space norm.

The classical model space X is the Paley-Wiener space $\mathcal{B}_\Omega$ of signals $f \in L^2(\mathbb{R})$ bandlimited to $[-\Omega/2, \Omega/2]$, *i.e.*, such that $\operatorname{supp}(\hat{f}) \subset [-\Omega/2, \Omega/2]$. Here we normalize the Fourier transform $\hat{f}$ of f as $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi it\xi}\, dt$ whenever the integral is defined. Signals in $\mathcal{B}_\Omega$ enjoy a particularly simple sampling property [14], [13], [18]:

Theorem 1 *(Classical Sampling Theorem) If $f \in \mathcal{B}_\Omega$ and $2T\Omega > 1$, then*

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) \mathit{sinc}\,(Tt - k) \tag{1}$$

where $\mathit{sinc}\,(t) = \dfrac{\sin \pi t}{\pi t}$ is the cardinal sine function and the convergence is both pointwise and in L^2-norm.

$\mathcal{B}_\Omega$ is a reasonable model for many phenomena. On the one hand, many natural and synthetic signal generators can only *output* slowly varying signals. On the other, many natural and synthetic processors can only *process* bandlimited signals. For example the human ear cannot process/amplify frequencies beyond, say, 20 kHz. The Classical Sampling Theorem is fundamental in digital signal processing since it provides a means of converting an analog signal f to a digital signal $\{f(\frac{k}{T})\}_{k=-\infty}^{\infty}$ and of reconstructing f uniquely and continuously within $\mathcal{B}_\Omega$ via (1). However, for many applications, different signal models are required, for example, for modelling the effect of real acquisition (sampling) devices, or for improved numerical performance of discrete processing algorithms. While the sinc functions are well-localized in frequency, they are poorly localized in time. In fact, their Heisenberg product $\|xf\|_2\|\xi\hat{f}\|_2$ is infinite, making them poor candidates for basis functions with which to implement post-sampling digital processing, *cf.*, [2]. In lieu of the sinc functions, we might consider as a basic signal model linear combinations of shifts of a function φ having better joint time-frequency localization, or with compact support, or any number of useful properties. Then, in analogy with $\mathcal{B}_\Omega$ one considers as a model space

$$V(\varphi) = \{f = \sum_k c_k\varphi(\cdot - k);\ c_k \in l^2(\mathbb{Z})\} \subset L^2(\mathbb{R}). \tag{2}$$

It is important to point out here that, although our main goal is to find conditions under which $f \in V(\varphi)$ can be recovered from samples of f, φ itself will not typically be a cardinal function, so the c_k will not be the samples of f. Therefore our problem can be restated as one of finding a *sampling function* S_φ, or a collection of sampling functions in $V(\varphi)$, such that f can be expressed as a superposition of such functions having sample values as coefficients. A property of sinc functions which tends not to be used in *classical* signal processing algorithms, but which goes a long way towards explaining the type of errors that the hypothesis of bandlimitedness incurs when it is not valid, is that $\operatorname{sinc}(t)$ is a *scaling function*. Surprisingly, the scaling property can also play a significant role in validating sampling methods in shift-invariant spaces.

Results of the form (1) are *uniform* sampling results in that the samples are taken at uniformly distributed points. The sampling of bandlimited signals at non-uniformly spaced points has a well-developed theory and its $V(\varphi)$-counterparts have also been studied (see [1] for a review), but in this paper we consider uniform and semi-uniform or *staggered* sampling only. We restrict to the situation where our model space is of the form (2) and we will call such a space a principal shift-invariant (PSI) space. It is also natural to consider finitely generated shift-invariant spaces, just as it is natural to consider sampling issues for such spaces in higher dimensions. However, these variations bring in nontrivial complications and might obscure the basic focus of developing techniques for validating various sampling schemes.

Note that $f(t) \in V(\varphi)$ implies that $f(t-k) \in V(\varphi)$ for all integers k. In the case where $\Omega = T = 1$, the Classical Sampling Theorem says that $f \in \mathcal{B}_\Omega$ can be continuously recovered from its samples, since the space is generated by the integer shifts of an orthogonal cardinal interpolating function, that is, a continuous function $S \in L^2(\mathbb{R})$ such that $S(k) = \delta_k$ ($k \in \mathbb{Z}$) and $\langle S, S(\cdot - l)\rangle = \delta_l$ ($l \in \mathbb{Z}$). In this situation, it is clear that the samples $f(k)$ of f are the same as the coefficients c_k in the orthogonal expansion $f(t) = \sum_k c_k S(t-k)$.

An orthogonal generator φ of $V(\varphi)$ will rarely be a cardinal interpolating function. The next best thing is the existence of a cardinal interpolating function $S = S_\varphi \in V(\varphi)$ that is no longer orthogonal to its integer shifts. We will say that *integer sampling in $V(\varphi)$ is possible* if the mapping $f \mapsto \{f(k)\}$ is bounded and continuously invertible from $V(\varphi)$ to $l^2(\mathbb{Z})$. We might ask also that for fixed $0 < t_0 < 1$, the mapping $f \mapsto \{f(t_0 + k)\}$ is bounded and continuously invertible from $V(\varphi)$ to $l^2(\mathbb{Z})$. Under the circumstances, we say that *translated integer sampling* in $V(\varphi)$ is possible.

Simple examples show that integer sampling does not always provide a bounded and continuously invertible operator. The first problem is to determine conditions on φ which guarantee that integer sampling is possible. When it is, there remains the second problem of determining the relationship between the PSI coefficients $\{c_k\}$ and samples $\{f(k)\}$ of f. For example, when φ is compactly supported, it seems reasonable to expect that the coefficients c_k should be locally determined, that is, should depend only on nearby samples. In fact, if φ is continuous, this is never the case, although it is often possible to estimate the matrix that sends $\{c_k\}$ to $\{f(k)\}$, [2]. One way of rectifying this non-locality problem is to sample at a rate higher than once per unit time, that is, to oversample. Oversampling in this context was studied by Djokovic and Vaidyanathan [5], and we build on their work here. A second scheme suggested also by Djokovic and Vaidyanathan, one that preserves the average sampling rate of once per unit time but which can produce compactly supported sampling functions when φ is compactly supported, is that of *staggered sampling* which is non-uniform, but periodic.

This paper is organized as follows. In Section 2 we introduce the Zak transform, principal shift-invariant (PSI) spaces and multiresolution analysis and develop the machinery needed in later sections. Section 3 contains a brief review of some of the ideas surrounding the sampling of signals in PSI spaces. Here the work of Janssen [12] and Walter [16] on (translated) integer (or "critical") sampling is discussed, as is the work of Djokovic and Vaidyanathan [5] on staggered critical sampling and uniform oversampling. The theoretical difficulties that each scheme presents as well as the advantages and disadvantages each has over the other will be outlined. The validity of the schemes is dependent upon the invertibility of a certain matrix, or the non-vanishing of a certain Fourier series, or the co-primality of a collection of polynomials. In Section 4 we check the validity of the schemes of Section 3 by providing conditions on the generators φ that validate the required invertibility, non-vanishing or co-primality conditions. For this purpose, common assumptions on φ such as compactness of support, bandlimitedness, orthogonality of shifts, and/or scale invariance are shown to be sufficient. Wherever possible, the dependence of adequate sampling rates on the length of the support of the generator, or its Fourier transform, are described.

2 Preliminaries

2.1 Zak Transform

The main tool used in the analysis of uniform and staggered sampling is the Zak transform. We outline here its relevant properties. For a more detailed discussion, including some history and applications to signal processing, the reader is referred to Janssen's excellent tutorial [11]. An account of the role of the Zak transform in Gabor analysis can be found in Daubechies' book [4] and the tutorial by Heil and Walnut [10].

Given $f \in \mathcal{S}(\mathbb{R})$, the Schwarz space of rapidly decreasing functions, and $t, \xi \in \mathbb{R}$, let the Zak transform Zf of f at the point (t, ξ) in *phase space* be given by

$$Zf(t,\xi) = \sum_k f(t+k)e^{2\pi i k\xi}.$$

The connection with the Poisson summation formula is apparent from the definition. Indeed, if $f \in \mathcal{S}(\mathbb{R})$, the Poisson summation formula may be written $Zf(t,\xi) = e^{-2\pi i t\xi} Z\hat{f}(-\xi, t)$. The Zak transform is quasiperiodic in the sense that

$$Zf(t+k,\xi+l) = e^{-2\pi il\xi} Zf(t,\xi) \qquad (l,k \in \mathbb{Z}).$$

Consequently, the values of Zf on the square $Q = [0,1) \times [0,1)$ determine the values of f on all of phase space and we think of Q as the domain of Zf. A simple consequence of quasiperiodicity, and of crucial importance to us, is the fact that if $Zf(t,\xi)$ is continuous in both variables, then Zf has a zero in Q [10].

The Zak transform is unitary in the sense that if $f, g \in \mathcal{S}(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt = \int_0^1 \int_0^1 Zf(t,\xi)\overline{Zg(t,\xi)}\, dt\, d\xi$$

and hence Z is a unitary mapping from $L^2(\mathbb{R})$ to

$$\mathcal{Z} = \left\{ F : \mathbb{R}^2 \to \mathbb{C} : \begin{array}{c} F \text{ is measurable, } F(t+k,\xi+l) = e^{-2\pi ik\xi} F(t,\xi)\ (k,l \in \mathbb{Z}) \\ \text{and } \int_0^1 \int_0^1 |F(t,\xi)|^2\, dt\, d\xi < \infty \end{array} \right\}. \tag{3}$$

The inversion formula is particularly simple: $\int_0^1 Zf(t,\xi)\, d\xi = f(t)$ whenever the integral converges. It will certainly do so for $f \in \mathcal{S}(\mathbb{R})$, and a limiting argument is required to deal with $f \in L^2(\mathbb{R})$.

It will be important for us, particularly when dealing with oversampled data, to extend the definition of the Zak transform from phase space $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \times \mathbb{C}$. As such, we define the complexified Zak transform $Z_{\mathbb{C}}f(x,z)$ for $x \in \mathbb{R}$, $z \in \mathbb{C}$ as the Laurent series

$$Z_{\mathbb{C}}f(t,z) = \sum_k f(t+k) z^k$$

whenever the sum converges. It is clear that $Z_{\mathbb{C}}f(t,z)$ is just the z-transform of the sequence of samples $\{f(t+k)\}_k$. When z is restricted to the unit circle, we recover the previous definition of the Zak transform, that is, $Z_{\mathbb{C}}f(t,e^{2\pi i\xi}) = Zf(t,\xi)$. Also, when f is supported on $[0,M]$ (M a positive integer) then $Z_{\mathbb{C}}f(t,z)$ is a polynomial in z of degree at most $M-1$.

2.2 Principal Shift-Invariant Spaces

A principal shift-invariant (PSI) space is a pair (V,φ) with V a closed subspace of $L^2(\mathbb{R})$ and $\varphi \in V$ with the property that the collection $\{\varphi(\cdot - k)\}_k$ is an orthonormal basis for V. Actually it is enough to assume that $\{\varphi(\cdot - k)\}_k$ is a Riesz basis for V, meaning that there exist constants $0 < A \le B < \infty$ such that $A\|c\|_{l^2} \le \|\sum_k c_k\varphi(\cdot - k)\|_{L^2} \le B\|c\|_{l^2}$ for all sequences $\{c_k\}_k \in l^2(\mathbb{Z})$. Indeed, the orthogonality condition on the integer shifts of φ may be written on the Fourier side as $\sum_k |\hat{\varphi}(\xi+k)|^2 = 1$ for a.e. ξ, while the Riesz basis condition takes the form $0 < A \le \sum_k |\hat{\varphi}(\xi+k)|^2 \le B < \infty$ for a.e. ξ.

The Shannon PSI space is the pair (V_S, φ_S) where $V_S = \mathcal{B}_1$ is nothing other than the Paley-Wiener space of L^2-signals bandlimited to $[-\frac{1}{2}, \frac{1}{2}]$ and $\varphi_S(t) = \frac{\sin \pi t}{\pi t} = \operatorname{sinc}(t)$. The Classical Sampling Theorem expresses each $f \in V_S$ as $f(t) = \sum_k f(k)\varphi_S(t-k)$. In this case $Z\varphi_S(t,\xi) = e^{2\pi it([\xi+\frac{1}{2}]-\xi)}$ which is discontinuous along the *horizontal* line $\xi = 1/2$.

As a second example, consider the space of functions V_H of signals $f \in L^2(\mathbb{R})$ with the property that f is constant on each interval $[k,k+1)$ ($k \in \mathbb{Z}$) with $\varphi_H(t) = \chi_{[0,1)}(t)$. The pair (V_H, φ_H) is called the Haar PSI space. Since each $f \in V_H$ is continuous except at the integers, it is clear that if $0 < t_0 < 1$, then samples taken at $t_0 + \mathbb{Z}$ are well-defined and determine f via

$$f(t) = \sum_k f(t_0+k)\varphi_H(t-k),$$

a sampling result for signals in V_H. It is worth noting that $Z\varphi_H(t,\xi) = e^{-2\pi i[t]\xi}$ is discontinuous along the *vertical* line $t = 0$.

This is an extreme pair of examples, each illustrating different features of sampling in PSI spaces. In the Shannon case, f is recovered from its integer samples while, in the Haar case, f is recovered from its values at integer shifts of any nonintegral t_0 although the value $f(t_0+k)$ is really better thought of as an average rather than as a pointwise value, since this is how it arises when V_H is thought of as the image of an orthogonal projection. We are interested in intermediate examples. Except in these extreme cases, however, the V-coefficients of $f \in V$, that is, the coefficients c_k in the expansion $f(t) = \sum_k c_k \varphi(t-k)$, will not be integer or translated integer samples of f.

It is worth pointing out here that both the Shannon function and the Haar function are scaling functions. The Haar function satisfies

$$\varphi_H(t) = \varphi_H(2t) + \varphi_H(2t-1)$$

while

$$\varphi_S(t) = \varphi_S(2t) + \sum_l \frac{(-1)^l}{\pi(l+\frac{1}{2})} \varphi_S(2t-2l-1).$$

On the surface this has nothing to do with the fact that functions in the corresponding PSI spaces can be recovered in particularly simple ways from their shifted values. However, we shall see that when the generator φ of a PSI space is a scaling function, that is, when it satisfies a relationship of the form

$$\frac{1}{2}\varphi\left(\frac{t}{2}\right) = \sum_k h_k \varphi(2t-k) \tag{4}$$

then it is possible to say things about the relationship between sample values of f and V-coefficients of f that could not be inferred otherwise. Besides this, when φ is a scaling function one has the ability to bring the structure of a multiresolution analysis of $L^2(\mathbb{R})$ to bear on issues of processing a given digitized/sampled signal in V. When such processing is desired it is even more important that a precise relationship between sample values and V-coefficients is understood. We will return to this issue in more detail in [9].

3 Sampling in Shift-Invariant Spaces

In this section we review some work in the field of sampling in PSI spaces. Although several authors give conditions that ensure the validity of the sampling procedures they propose in particular cases, those conditions have proved difficult to check over large classes of generators.

3.1 Critical Sampling

The most basic sampling problem for a PSI space is that of recovering $f \in V(\varphi)$ from its samples $\{f(k)\}_{k=-\infty}^{\infty}$. Equivalently, one asks whether there is an analog of the Classical Sampling Theorem for $V(\varphi)$ – in particular whether there exists a sampling function $S \in V(\varphi)$ such that

- S is cardinal, i.e., $S(k) = \delta_k$ $(k \in \mathbb{Z})$,
- $f(t) = \sum_k c_k \varphi(t-k) = \sum_l f(l) S(t-l)$, and
- the mapping $\{c_k\}_k \mapsto \{f(k)\}_k$ from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$ thought of, firstly, as the space of coefficients of functions in $V(\varphi)$ and secondly as the space of integer samples of signals in $V(\varphi)$, is bounded and continuously invertible.

Janssen [12] was the first to note the special role played by the Zak transform in this setting. If $f(t) = \sum_k c_k \varphi(t-k) \in V(\varphi)$ $(\{c_k\}_k \in l^2(\mathbb{Z}))$, then evaluating both sides of this equation at $l \in \mathbb{Z}$ gives $f(l) = \sum_k c_k \varphi(l-k)$, i.e., the integer samples of f are obtained by filtering the coefficients c_k with the sequence $\{\varphi(l)\}_l$. Forming the Fourier series of both sides of this equation yields $F(\xi) = C(\xi)\Phi(\xi)$ where F, C and Φ

are the Fourier series of the sequences $f(l)$, c_l and $\varphi(l)$ respectively. Walter [16] noticed that if $\inf_\xi |\Phi(\xi)| \neq 0$, then

$$C(\xi) = \frac{F(\xi)}{\Phi(\xi)} \tag{5}$$

and since $A(\xi) = \Phi(\xi)^{-1} \in L^\infty([0,1])$ we may compute the Fourier coefficients of both sides of (5) for C to obtain

$$c_k = \sum_l f(l) a_{k-l},$$

where $\{a_l\}_l$ are the Fourier coefficients of the periodic function A. Hence,

$$f(t) = \sum_k c_k \varphi(t-k) = \sum_k \sum_l f(l) a_{k-l} \varphi(t-k) = \sum_l f(l) S(t-l) \tag{6}$$

where $S(t) = \sum_k a_k \varphi(t-k) \in V(\varphi)$. Walter provides examples of scaling functions φ for which the assumption $\inf_\xi |\Phi(\xi)| > 0$ is valid (the Daubechies scaling functions φ_ν supported on $[0,3]$) and examples where it is not (the quadratic B-splines, in which case $\Phi(\frac{1}{2}) = 0$).

Janssen [12] introduced the notion of translated integer sampling – determining a signal f from its translated integer samples $\{f(t_0+k)\}_k$ for some fixed t_0. If $f(t) = \sum_k c_k \varphi(t-k) \in V(\varphi)$, then taking the Zak transform of both sides gives

$$Zf(t,\xi) = C(\xi) Z\varphi(t,\xi).$$

If there exists t_0 such that $\inf_\xi |Z\varphi(t_0,\xi)| \neq 0$, then $(Z\varphi(t_0,\xi))^{-1} \in L^\infty([0,1])$ and we may write $C(\xi) = Zf(t_0,\xi)/Z\varphi(t_0,\xi)$. Hence

$$\begin{aligned} f(t) &= \sum_k \left(\int_0^1 C(\xi) e^{-2\pi i k \xi}\, d\xi \right) \varphi(t-k) \\ &= \sum_k \left(\int_0^1 \frac{Zf(t_0,\xi)}{Z\varphi(t_0,\xi)} e^{-2\pi i k \xi}\, d\xi \right) \varphi(t-k) \\ &= \sum_k \sum_l f(t_0+l) \left(\int_0^1 \frac{e^{2\pi i (l-k)\xi}}{Z\varphi(t_0,\xi)}\, d\xi \right) \varphi(t-k) = \sum_l f(t_0+l) S_{t_0}(t-l) \end{aligned} \tag{7}$$

where $S_{t_0}(t) = \int_0^1 \frac{Z\varphi(t,\xi)}{Z\varphi(t_0,\xi)}\, d\xi = \sum_k b_k \varphi(t-k)$ in which $b_k = \int_0^1 \frac{e^{-2\pi i k\xi}}{Z\varphi(t_0,\xi)}\, d\xi \in l^2(\mathbb{Z})$. S_{t_0} is t_0-cardinal in the sense that $S_{t_0}(t_0+k) = \delta_k$. Janssen [12] computes these sampling functions for several examples. This scheme also provides flexibility in that the shift t_0 may be thought of as a parameter against which the aliasing performance of the scheme can be optimized [12], [7], [9].

The condition that the shifts of $\varphi \in L^2(\mathbb{R})$ form a Riesz basis for $V(\varphi)$ is insufficient to ensure that integer sampling is valid in $V(\varphi)$. In the case of the quadratic B-spline $\varphi = B_2(t)$, $\Phi(\frac{1}{2}) = 0$ and, since Φ is a polynomial, $1/\Phi \notin L^2$ so integer sampling in $V(B_2)$ is invalid. But, even worse, the space $V(\varphi)$ generated by the shifts of φ may not admit translation sampling *for any translation*! This occurs when φ has compact support and $\inf_\xi |Z\varphi(t,\xi)| = 0$ for all t. In fact, we can construct a continuous, compactly supported function supported on $[0,2]$ with orthogonal integer translates such that $Z\varphi(t,t) = 0$ for all t. The key observation in the construction is that the space $\mathcal{Z}$ introduced in (3) as the range of the Zak transform $Z : L^2(\mathbb{R}) \to \mathcal{Z} \subset L^2(Q)$ is invariant under multiplication by functions in $A(\mathbb{T})$ (the space of Fourier series of l^1-sequences) in either the t or ξ variables. If $P(\xi) = \sum_k p_k e^{2\pi i k\xi} \in A(\mathbb{T})$, then $P(\xi) Zf(t,\xi) = Zf_P(t,\xi)$ where $f_P(t) = \sum_k p_k f(t-k) \in L^2(\mathbb{R})$ and if $Q(t) = \sum_l q_l e^{2\pi i l t} \in A(\mathbb{T})$, then $Q(t) Zf(t,\xi) = Zf^Q(t,\xi)$ where $f^Q(t) = Q(t) f(t) \in L^2(\mathbb{R})$. Now take f to be $f(t) = (\sum_k c_k e^{2\pi i k t}) \chi_{[0,1]}(t)$. To ensure that f is continuous, we will need $\{c_k\}_k \in l^1(\mathbb{Z})$ and $f(0^+) = \sum_k c_k = 0$. Let $g(t) = e^{2\pi i t} f(t) - f(t-1)$. Then g is supported on $[0,2]$ and $Zg(t,\xi) = (e^{2\pi i t} - e^{2\pi i \xi}) Zf(t,\xi)$ vanishes on the diagonal $\xi = t$. Orthogonality of the shifts of g may be obtained by placing more conditions on the sequence $\{c_k\}_k$ defining f. In particular, if $f(t) = i\sin 2\pi t\chi_{[0,1)}(t)$ and

$$g(t) = e^{2\pi it}f(t) - f(t-1) = i\sin 2\pi t \begin{cases} e^{2\pi it} & \text{if } 0 \le t < 1 \\ -1 & \text{if } 1 \le t \le 2 \\ 0 & \text{else,} \end{cases}$$

then g is continuous, supported on $[0,2]$, has orthogonal (integer) shifts, satisfies $\hat{g}(0) = \frac{1}{2}$, $\|g\|_2 = 1$ and $Zg(t,t) = 0$ for all t. The space $V(g)$ then admits translated integer sampling at no point.

Until now, no criterion has been given which ensures the efficacy of shifted sampling. The question which naturally arises from this discussion, and which has been outstanding since the publication of Janssen's paper [12], is whether a continuous orthogonal generator φ satisfying a scaling relation (4) admits translated integer sampling at some point, that is, $\inf_\xi |Z\varphi(t_0,\xi)| > 0$ for some t_0. The answer is still unknown, although there is some evidence to suggest that when φ is also supported on $[0,2^J]$, $\inf_\xi |Z\varphi(\frac{l}{2^J},\xi)| > 0$ for some $0 \le l \le 2^J - 1$.

Even when (translated) integer sampling is available, its implementation can have undesirable consequences. Suppose φ is a continuous scaling function supported on $[0,M]$ and $\inf_\xi |\Phi(\xi)| > 0$. Then the sampling function S which appears in (6) will not be compactly supported unless $|\Phi(\xi)|$ is constant, which implies (since Φ is a trigonometric polynomial with $\Phi(0) = 1$) that $\Phi(\xi) = e^{2\pi iP\xi}$ for some integer $1 \le P \le M-1$, that is, $\varphi(k) = \delta_{k-P}$, a cardinality condition. Xia and Zhang [17] show that there is no such continuous φ, although the Haar scaling function provides a discontinuous example. In the case of shifted integer sampling, compactness of the support of the sampling function S_{t_0} is equivalent to the translated cardinality condition $\varphi(t_0 + k) = \delta_{k-P}$. When $t_0 = 2^{-J}$ $(J \ge 1)$, no continuous compactly supported scaling function can satisfy this property. For general t_0, the question of the existence of continuous compactly supported t_0-cardinal scaling functions remains open.

3.2 Staggered Sampling

In [5], Djokovic and Vaidyanathan introduced a scheme that we call *staggered sampling* (see also [15]). This means that sampling points $t_1, \ldots t_N$ are chosen non-uniformly – usually within the support of φ – and one shifts this set of points uniformly to produce a semi-uniform sampling set of the form $\Lambda = \{t_1, \ldots, t_N\} + N\mathbb{Z}$. In this way the average sampling rate $\lim_{T\to\infty} \frac{1}{2T}\#\{\Lambda \cap [-T,T]\}$ is critical, i.e, is equal to 1, but the sampling set itself is best thought of as a union of N disjoint uniform sampling sets, each of which has rate $1/N$ times the critical rate. The practical significance of such a scheme is that it allows sampling with a single device at a "high rate" to be replaced by sampling with staggered devices at a lower rate. The effect of staggering, ostensibly, is to avoid any phase coupling among the samples that might occur when the signal is assumed to lie in a PSI space. To recover f from its staggered samples one requires N sampling functions in $V(\varphi)$ – one for each of the uniform subsets. There are several such schemes. We outline just two of them here.

SS1: Let φ be supported on $[0,M]$. Choose $0 \le t_0 < t_1 < \cdots < t_{M-1} < 1$ and take samples at $\{t_0, t_1, \ldots, t_{M-1}\} + M\mathbb{Z}$. Let $f(t) = \sum_k c_k\varphi(t+k) \in V(\varphi)$. The staggered samples of f then satisfy

$$f(t_j + mM) = \sum_k c_k\varphi(t_j + mM + k) = \sum_{k=0}^{M-1} c_{k-mM}\varphi(t_j + k) = \sum_{k=0}^{M-1} M^\varphi_{jk}c_{k-mM}$$

where M^φ_{jk} is the $(j,k)^{\text{th}}$ entry in the $M \times M$ matrix M^φ given by

$$M^\varphi_{jk} = \varphi(t_j + k) \ (0 \le j,k \le M-1). \tag{8}$$

If M^φ is invertible, we have $c_{j-mM} = \sum_{l=0}^{M-1} (M^\varphi)^{-1}_{jl} f(t_l + mM)$ and, therefore,

$$\begin{aligned} f(t) &= \sum_k c_k \varphi(t+k) = \sum_m \sum_{j=0}^{M-1} c_{j-mM} \varphi(t+j-mM) \\ &= \sum_m \sum_{l=0}^{M-1} f(t_l + mM) \sum_{j=0}^{M-1} (M^\varphi)^{-1}_{jl} \varphi(t+j-mM) \\ &= \sum_m \sum_{l=0}^{M-1} f(t_l+mM) S_l(t-mM) \end{aligned}$$

where $S_l(t) = \sum_{j=0}^{M-1} (M^\varphi)^{-1}_{jl} \varphi(t+j) \in V(\varphi)$ is supported on $[1-M, M]$.

In several cases, it is possible to check that M^φ is invertible, and examples of this type are considered in [5]. Specifically, it is shown in [5] that when φ is the quadratic B-spline supported on $[0,3]$– a situation in which integer sampling is not valid – $\det M^\varphi$ is a Vandermonde determinant and hence does not vanish as long as the sampling nodes t_0, t_1, t_2 are distinct. Similar results valid for higher-order splines are also available. However, no general conditions on the invertibility of M^φ are given in [5], as little can be inferred without assuming that φ satisfies some additional condition such as a scaling relation. In Section 4 we deduce the validity of SS1 for the family φ_ν of Daubechies' 4-coefficient scaling functions supported on $[0,3]$ and give an equivalent condition for the invertibility of M^φ involving the operator T_H in (21).

More general matrix invertibility conditions corrseponding to different sampling lattices are possible, as following sampling scheme demonstrates.

SS2: We present here a slightly simplified version of a second staggered sampling scheme from [5]. Suppose φ is supported in $[0, M]$. Choose $2M-1$ points $0 \le t_1 < t_2 < \cdots < t_{2M-1} < 1$ and sample $f \in V(\varphi)$ at $\{t_1, t_2, \ldots t_{2M-1}\} + (2M-1)\mathbb{Z}$. In [5] this is replaced by $NM-1$ points in $[0, M]$ and their translates by multiples of $NM-1$. If $f(t) = \sum_k c_k \varphi(t-k)$ then, after sampling we have for each integer l,

$$f(t_p + l(2M-1)) = \sum_k c_k \varphi(t_p + l(2M-1) - k) = \sum_k c_{k+l(2M-1)} \varphi(t_p - k). \tag{9}$$

Since φ is supported in $[0, M]$, the only terms contributing to the sum are those for which $1-M \le k \le M-1$. Define vectors $\mathbf{f}^{(l)}$, $\mathbf{c}^{(l)} \in \mathbb{C}^{2M-1}$ and a $(2M-1)\times(2M-1)$ matrix N^φ by $\mathbf{f}^{(l)}_p = f(t_p + l(2M-1))$, $\mathbf{c}^{(l)}_k = c_{k+l(2M-1)}$ and

$$N^\varphi_{pk} = \varphi(t_p - k) \quad (1-M \le k \le M-1, 1 \le p \le 2M-1). \tag{10}$$

Equation (9) may be written as the matrix equation $\mathbf{f}^{(l)} = N^\varphi \mathbf{c}^{(l)}$. If N^φ is invertible, the coefficients c_k may be determined *locally* by $\mathbf{c}^{(l)} = (N^\varphi)^{-1} \mathbf{f}^{(l)}$, meaning that

$$c_{k+l(2M-1)} = \sum_{r=1}^{2M-1} (N^\varphi)^{-1}_{kr} f(t_r + l(2M-1)).$$

Consequently,

$$\begin{aligned} f(t) &= \sum_l \sum_{k=1-M}^{M-1} c_{k+l(2M-1)} \varphi(t-k-l(2M-1)) \\ &= \sum_l \sum_{k=1-M}^{M-1} \sum_{r=1}^{2M-1} (N^\varphi)^{-1}_{kr} f(t_r + l(2M-1)) \varphi(t-k-l(2M-1)) \\ &= \sum_l \sum_{r=1}^{2M-1} f(t_r + l(2M-1)) S_r(t - l(2M-1)) \end{aligned}$$

where the sampling functions $S_r(t)$ are defined by

$$S_r(t) = \sum_{k=1-M}^{M-1} (N^\varphi)^{-1}_{kr} \varphi(t-k) \in V(\varphi)$$

and are supported on $[1-M, 2M-1]$.

In Section 4, scheme SS2 is validated for the scale φ_ν $(-1 < \nu < 0)$ of Daubechies scaling functions supported on $[0,3]$, and a general sufficient condition on scaling functions φ is given to ensure the invertibility of N^φ and therefore the validity of SS2. In summary, staggered sampling is a form of critical, multiply shifted sampling in which f is reconstructed from its staggered samples with multiple sampling functions.

3.3 Oversampling

Another technique for overcoming the problems associated with integer sampling or shifted integer sampling is to *oversample*, that is, to sample at a rate higher than once per shift. The cost of a higher sampling rate is often compensated by improved properties of the sampling functions. For example, they can inherit compactness of support from the generator φ. Furthermore, added flexibility can be exploited to improve performance of implementations, *e.g.* [7], [8]. We present here two possible oversampling schemes and leave the validation of their conditions until Section 4.

OS1: This is the most primitive of the oversampling schemes. We assume that φ generates a shift-invariant space $V(\varphi)$, $L \geq 2$ is an integer and $0 \leq t_0 < t_1 < \cdots < t_{L-1} < 1$ are such that

$$\sum_{m=0}^{L-1} |Z\varphi(t_l, \xi)|^2 \geq c > 0 \tag{11}$$

for all ξ. If $f(t) = \sum_k c_k \varphi(t-k) \in V(\varphi)$ then $Zf(t,\xi) = C(\xi) Z\varphi(t,\xi)$. Putting $t = t_l$, multiplying both sides by $\overline{Z\varphi(t_l,\xi)}$ and summing over l gives

$$\sum_{l=0}^{L-1} Zf(t_l,\xi)\overline{Z\varphi(t_l,\xi)} = C(\xi) \sum_{j=0}^{L-1} |Z\varphi(t_j,\xi)|^2.$$

Dividing both sides by the non-vanishing Fourier series on the left-hand-side of (11) gives

$$C(\xi) = \frac{\sum_{l=0}^{L-1} Zf(t_l,\xi)\overline{Z\varphi(t_l,\xi)}}{\sum_{j=0}^{L-1} |Z\varphi(t_j,\xi)|^2} = \frac{\sum_{l=0}^{L-1} \sum_m f(t_l+m) e^{2\pi i m\xi} \overline{Z\varphi(t_l,\xi)}}{\sum_{j=0}^{L-1} |Z\varphi(t_j,\xi)|^2}$$

and therefore f may be reconstructed from its samples via

$$\begin{aligned}
f(t) &= \sum_k c_k \varphi(t-k) \\
&= \sum_k \int_0^1 \frac{\sum_{l=0}^{L-1} \sum_m f(t_l+m) e^{2\pi i m\xi} \overline{Z\varphi(t_l,\xi)}}{\sum_{j=0}^{L-1} |Z\varphi(t_j,\xi)|^2} e^{-2\pi i k\xi}\, d\xi \varphi(t-k) \\
&= \sum_{l=0}^{L-1} \sum_m f(t_l+m) \int_0^1 \frac{Z\varphi(t,\xi)\overline{Z\varphi(t_l,\xi)}}{\sum_{j=0}^{L-1} |Z\varphi(t_j,\xi)|^2} e^{2\pi i m\xi}\, d\xi \\
&= \sum_{l=0}^{L-1} \sum_m f(t_l+m) \int_0^1 \frac{Z\varphi(t-m,\xi)\overline{Z\varphi(t_l,\xi)}}{\sum_{j=0}^{L-1} |Z\varphi(t_j,\xi)|^2}\, d\xi = \sum_{l=0}^{L-1} \sum_m f(t_l+m) S_l(t-m)
\end{aligned}$$

where

$$S_l(t) = \sum_k \left\{ \int_0^1 \frac{\overline{Z\varphi(t_l,\xi)} e^{-2\pi i k\xi}}{\sum_{j=0}^{L-1} |Z\varphi(t_j,\xi)|^2} \, d\xi \right\} \varphi(t-k) \in V(\varphi). \tag{12}$$

In Section 4 we give conditions under which the assumption (11) is valid along with estimates of the sampling rate L.

OS2: A elegant scheme can be constructed from information on zeroes of the Zak transform. The assumption made on φ is that it is compactly supported on $[0, M]$ and that there exists an integer $L \geq 2$ and $0 \leq t_0 < t_1 < \cdots < t_{L-1} < 1$ such that the polynomials $\{Z_{\mathbb{C}}\varphi(t_l, z)\}_{l=0}^{L-1}$ are co-prime. As we shall see, a weak continuity condition on φ and orthogonality of its integer shifts imply the existence of L. Estimates of L require more sophisticated methods and more restrictive conditions. Note that since $\{Z_{\mathbb{C}}\varphi(t_l, z)\}_{l=0}^{L-1}$ are co-prime and have degree less than or equal to $M-1$, Euclid's algorithm gives polynomials $P_0, P_1, \ldots P_{L-1}$ of degree less than or equal to $M-2$ such that

$$\sum_{l=0}^{L-1} P_l(z) Z_{\mathbb{C}}\varphi(t_l, z) = 1 \tag{13}$$

for all $z \in \mathbb{C}$. Now suppose $f(t) = \sum_k c_k \varphi(t-k) \in V(\varphi)$. Multiplying both sides of (13) by $C(z)$ gives

$$C(z) = \sum_{l=0}^{L-1} P_l(z) C(z) Z_{\mathbb{C}}\varphi(t_l, z) = \sum_{l=0}^{L-1} P_l(z) Z_{\mathbb{C}} f(t_l, z).$$

When $P_l(z) = \sum_{j=0}^{M-2} p_l^j z^j$, the scaling coefficients of f are recovered by

$$c_k = \sum_{l=0}^{L-1} \int_0^1 P_l(\xi) Z f(t_l, \xi) e^{-2\pi i k\xi} \, d\xi = \sum_{l=0}^{L-1} \sum_{j=0}^{M-2} p_l^j f(t_l + k - j)$$

and consequently f may be reconstructed from its samples $f(t_l + k)$, $0 \leq l \leq L-1$, $k \in \mathbb{Z}$ by

$$\begin{aligned} f(t) &= \sum_k \sum_{l=0}^{L-1} \sum_{j=0}^{M-2} p_l^j f(t_l + k) \varphi(t + j - k) \\ &= \sum_k \sum_{l=0}^{L-1} f(t_l + k) \sum_{j=0}^{M-2} p_l^j \varphi(t + j - k) = \sum_k \sum_{l=0}^{L-1} f(t_l + k) S_l(t-k) \end{aligned}$$

where $S_l(t) = \sum_{j=0}^{M-2} p_l^j \varphi(t+j) \in V(\varphi)$ is supported on $[2-M, M]$.

OS3: Schemes involving polynomials of lower degree, and hence sampling functions of smaller support, are possible at the expense of higher sampling rates. Suppose $0 \leq t_0 < t_1 < \cdots < t_{L-1} < 1$, φ is supported on $[0, M]$ and $\{Z_{\mathbb{C}}\varphi(t_l, \cdot)\}_{l=0}^{L-1}$ are co-prime. We seek polynomials $P_0, P_1, \ldots, P_{L-1}$ of degree less than or equal to D such that

$$\sum_{l=0}^{L-1} P_l(z) Z_{\mathbb{C}}\varphi(t_l, z) = 1 \tag{14}$$

for all $z \in \mathbb{C}$. Equating coefficients on both sides of (14) we see that we have a system of $M + D$ equations in the $N(D+1)$ coefficients p_l^j $(0 \leq l \leq L-1,\ 0 \leq j \leq D)$. The simplest case is that which arises when $D = 0$, so that we are trying to find $a_0, a_1, \ldots, a_{L-1} \in \mathbb{C}$ such that $\sum_{l=0}^{L-1} a_l Z_{\mathbb{C}}\varphi(t_l, z) = 1$ for all $z \in \mathbb{C}$. Equivalently, $\sum_{l=0}^{L-1} a_l \varphi(t_l + k) = \delta_k$. A solution exists precisely when $\mathbf{e}_0 = (1, 0, \ldots, 0)^t \in \mathrm{Ran}\,(P^{\varphi})$ where

P^φ is the matrix $P_{kl} = \varphi(t_l + k)$ $(0 \le l \le L-1,\ 0 \le k \le M-1)$. When $L = M$, we see that $P^\varphi = (M^\varphi)^t$ with M^φ the matrix appearing in the staggered sampling scheme SS1. Hence, in this case, OS3 is valid whenever SS1 is valid. If $\mathbf{e}_0 \in \operatorname{Ran}(P^\varphi)$, let $\mathbf{a} = (a_0, a_1, \ldots, a_{L-1})$ be any solution of $P^\varphi \mathbf{a} = \mathbf{e}_0$. With the a_j's so defined, we have for all $f(t) = \sum_k c_k \varphi(t-k) \in V(\varphi)$, $C(z) = \sum_{l=0}^{M-1} a_l Z_{\mathbb{C}} f(t_l, z)$ and hence $c_k = \sum_{l=0}^{M-1} a_l f(t_l + k)$. So

$$f(t) = \sum_k \left(\sum_{l=0}^{M-1} a_l f(t_l + k) \right) \varphi(t-k).$$

One of the attractive features of this scheme is that the sampling functions are just the scaling functions and the coefficients c_k are simply weighted samples of f taken on $[k, k+1)$. Also notice that

$$f(t) = \sum_{l=0}^{M-1} a_l \left(\sum_k f(t_l + k) \varphi(t-k) \right). \tag{15}$$

If f was bandlimited to $[-1/2, 1/2]$ and φ was the cardinal sine function, the inner sum in equation (15) would, by the Classical Sampling Theorem, be just $f(t_l + t)$. Although this is not the case for a general PSI space, a weighted average of the approximations $\sum_k f(t_n + k)\varphi(t-k)$ as in (15) does in fact reconstruct f provided $\mathbf{e}_0 \in \operatorname{Ran}(P^\varphi)$.

4 Validation of Sampling Schemes

In this section, conditions are imposed on generators φ which ensure the validity of the sampling schemes developed in Section 3. As expected, we show how strengthening the conditions on φ gives stronger statements about the validity, utility and efficiency of sampling schemes. These conditions include smoothness assumptions, compact support, bandlimitedness, orthogonality of integer shifts and, most importantly, the satisfaction of dilation equations. Because of the importance of scaling properties in the sequel, it will serve us to review several generic properties of a multiresolution analysis.

4.1 Multiresolution Analysis

A multiresolution analysis (MRA) of $L^2(\mathbb{R})$ is a nested sequence of closed subspaces $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$ and a function $\varphi \in V_0$ such that

- (V_0, φ) is a PSI space,
- $f(t) \in V_j \iff f(2t) \in V_{j+1}$,
- $\overline{\cup_j V_j} = L^2(\mathbb{R})$,
- $\cap_j V_j = \{0\}$.

Since φ generates V_0 in the sense that $\{\varphi(t-k)\}_k$ is an orthonormal basis for V_0, the second of the properties above implies that $\{2^{j/2}\varphi(2^j t - k)\}_k$ is an orthonormal basis for V_j. Also, since $V_{-1} \subset V_0$, φ must satisfy a scaling relationship (4) for some sequence $\{h_k\}_k$. In fact, the orthogonality of the integer shifts of φ gives us that $h_k = \frac{1}{2}\int_{-\infty}^{\infty} \varphi(\frac{t}{2})\overline{\varphi(t-k)}\, dt$. On the Fourier side, (4) may be written

$$\hat{\varphi}(2\xi) = H(-\xi)\hat{\varphi}(\xi) \tag{16}$$

where $H(\xi) = \sum_k h_k e^{2\pi i k \xi}$ is the Fourier series of $\{h_k\}_k$. To satisfy the density condition of an MRA, φ must satisfy $\int \varphi \neq 0$, and evaluating both sides of (16) at $\xi = 0$ gives $H(0) = 1$ from which we obtain $\hat{\varphi}(0) = \int \varphi = 1$. In the Haar case, $\varphi_H(t) = \chi_{[0,1)}(t)$, $h_0 = h_1 = \frac{1}{2}$ and $h_k = 0$ if $k \neq 0, 1$, so that

$H(\xi) = e^{\pi i \xi} \cos \pi\xi$. In the Shannon case $\hat{\varphi}_S(\xi) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi)$ and $H(\xi) = \chi_{[0,\frac{1}{4}]}(\xi) + \chi_{[\frac{3}{4},1]}(\xi)$ on $[0,1]$ is periodic with period 1.

There are several important conditions we impose on the low-pass filter H. The first of these is the quadrature mirror filter (QMF) condition

$$|H(\xi)|^2 + |H(\xi + \frac{1}{2})|^2 \equiv 1. \tag{17}$$

It will be important for us to extend this condition off the circle. Let $H_{\mathbb{C}}(z) = \sum_k h_k z^k$ $(z \in \mathbb{C})$ denote the z-transform of the sequence $\{h_k\}_k$ whenever the sum converges. Then the QMF condition (17) becomes

$$H_{\mathbb{C}}(z)\overline{H_{\mathbb{C}}(z^*)} + H_{\mathbb{C}}(-z)\overline{H_{\mathbb{C}}(-z^*)} = 1 \quad (z \neq 0) \tag{18}$$

where $z^* = \overline{z}^{-1}$. The other condition we might impose on H is the *τ-cycle condition*: there is no non-trivial cycle $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ in the unit circle $\mathbb{T}$ in the complex plane ($\tau\zeta_i \equiv \zeta_i^2 = \zeta_{i+1}$ with $\zeta_1 = \zeta_n$) such that $|H_{\mathbb{C}}(\zeta_i)| = 1$ for $1 \leq i \leq n$.

For each positive integer J, define $\varphi_J \in L^2(\mathbb{R})$ via its Fourier transform

$$\hat{\varphi}_J(\xi) = \prod_{j=1}^{J} H\left(\frac{-\xi}{2^j}\right) \chi_{[-2^{J-1},2^{J-1}]}(\xi).$$

From repeated application of equation (16) we would expect

$$\hat{\varphi}(\xi) = \lim_{J\to\infty} \hat{\varphi}_J(\xi) = \lim_{J\to\infty} \prod_{j=1}^{J} H(\frac{-\xi}{2^j}). \tag{19}$$

In fact, if H is a trigonometric polynomial satisfying the QMF condition (17), $H(0) = 1$ and φ is determined via (19) then the τ-cycle condition is equivalent to the orthogonality of the integer shifts of φ.

It was observed by Janssen [12] that the dilation equation (4) forces a self-similarity condition on the Zak transform of φ. Given a Laurent series $A(z)$ convergent at least on $\mathbb{T}$, we define the operator T_A which acts on Laurent series via

$$T_A f(z^2) = A(z)f(z) + A(-z)f(-z).$$

The restriction of T_A to functions on $\mathbb{T}$ (still denoted by T_A) is given by

$$T_A f(\xi) = A\left(\frac{\xi}{2}\right) f\left(\frac{\xi}{2}\right) + A\left(\frac{\xi}{2} + \frac{1}{2}\right) f\left(\frac{\xi}{2} + \frac{1}{2}\right). \tag{20}$$

With this notation, taking the Zak transforms of both sides of (4) yields

$$Z_{\mathbb{C}}\varphi(x,z) = T_H(Z_{\mathbb{C}}\varphi(2t,\cdot))(z), \qquad Z\varphi(t,\xi) = T_H(Z\varphi(2t,\cdot))(\xi). \tag{21}$$

Similar relations are satisfied by the Zak transform of the wavelet ψ associated to φ. In this paper our focus is on scaling functions rather than wavelets, but knowledge of $Z\psi$ is crucial in questions of wavelet design [3], [6], [9] and for the analysis of aliasing effects [7], [8], [9]. One way to interpret equation (21) is as follows: given the values of $Z\varphi$ on the line $t = t_0$, we can compute the values of $Z\varphi$ on the line $t = \frac{t_0}{2}$ with an application of the operator T_H. If we define M to be the operator of multiplication by z^{-1}, i.e., $Mf(z) = z^{-1}f(z)$ for Laurent series f, then the quasiperiodicity of Zak transforms may be described by $Z_{\mathbb{C}}f(t+1,z) = M(Z_{\mathbb{C}}f(t,\cdot))(z)$. Hence, for each dyadic rational $\frac{p}{2^j}$ $(p, j \in \mathbb{Z},\ j \geq 0)$, multiple applications of (21) give

$$Z_{\mathbb{C}}\varphi\left(\frac{p}{2^j}, z\right) = T_H^j(Z_{\mathbb{C}}\varphi(p,\cdot))(z) = T_H^j M^p \Phi_{\mathbb{C}}(z) \tag{22}$$

where $\Phi_{\mathbb{C}}(z) = Z_{\mathbb{C}}\varphi(0,z) = \sum_k \varphi(k)z^k$. Consequently, given Φ and H, (22) allows us to calculate (iteratively) the values of $Z_{\mathbb{C}}\varphi(t,z)$ for all z and all dyadic rationals t. See [6], [9] for applications of these ideas to the construction of compactly supported wavelets and scaling functions amenable to critical sampling, extrapolation etc.

Define an inner product on Laurent series $f(z)$, $g(z)$ by

$$\langle f, g\rangle = \frac{1}{2\pi i}\int_{\mathbb{T}} f(z)\overline{g(z)}\,\frac{dz}{z} = \int_0^1 f(e^{2\pi i\xi})\overline{g(e^{2\pi i\xi})}\,d\xi.$$

Then T_A has adjoint T_A^* given by

$$T_A^*g(z) = 2\overline{H_{\mathbb{C}}(z^*)}g(z^2); \quad T_A^*g(\xi) = 2\overline{H(\xi)}g(2\xi).$$

Also, observe that for all $g \in L^2(\mathbb{T})$,

$$\begin{aligned}
\|T_A^*g\|_2^2 &= 4\int_0^1 |A(\xi)|^2|g(2\xi)|^2\,d\xi \\
&= 4\left(\int_0^{1/2} |A(\xi)|^2|g(2\xi)|^2\,d\xi + \int_{1/2}^1 |A(\xi)|^2|g(2\xi)|^2\,d\xi\right) \\
&= 4\int_0^{1/2}\left[|A(\xi)|^2 + |A(\xi+\frac{1}{2})|^2\right]|g(2\xi)|^2\,d\xi,
\end{aligned}$$

from which we conclude that

$$2\inf_{\xi}\left[|A(\xi)|^2 + |A(\xi+\frac{1}{2})|^2\right]\|g\|_2^2 \le \|T_A^*g\|_2^2 \le 2\sup_{\xi}\left[|A(\xi)|^2 + |A(\xi+\frac{1}{2})|^2\right]\|g\|_2^2.$$

If H satisfies the QMF condition (17), we have $\|T_H^*g\|_2^2 = 2\|g\|_2^2$. In fact, $T_H T_H^* = 2\,\mathrm{Id}$ on $L^2(\mathbb{T})$ from which we see that $\frac{1}{2}T_H^*$ is a right inverse for T_H.

Now we investigate the kernel of the operator T_H. Suppose $f \in \mathrm{Ker}\,(T_H)$, i.e, $H_{\mathbb{C}}(z)f(z)+H_{\mathbb{C}}(-z)f(-z) = 0$ for all z. Then

$$\begin{pmatrix} f(z) \\ f(-z) \end{pmatrix} = \alpha(z)\begin{pmatrix} H_{\mathbb{C}}(-z) \\ -H_{\mathbb{C}}(z) \end{pmatrix}$$

for some function α. Since $f(z) = \alpha(z)H_{\mathbb{C}}(-z)$ and $f(-z) = -\alpha(z)H_{\mathbb{C}}(z)$, we have $\alpha(-z) = -\alpha(z)$, that is, α is an odd function on $\mathbb{C}$. Hence the kernel of T_H consists of those Laurent polynomials F such that $F(z) = \alpha(z)H_{\mathbb{C}}(-z)$ with α odd.

4.2 Validation of Staggered Sampling Techniques

Demonstrating that staggered sampling techniques are valid over a class of generators φ is a difficult matter. Djokovic and Vaidyanathan [5] consider particular examples of scaling functions such as the quadratic B-spline. In particular, they show that the matrix M_φ of SS1 is a Vandermonde matrix and hence its determinant is non-zero provided $t_0, t_1, \ldots, t_{M-1}$ are distinct. The same is true for B-splines of all orders [9]. Beyond that, little is known about the validity of staggered sampling except for particular examples of lattices and generators.

SS1: Here we check the validity of staggered sampling scheme SS1 for the Daubechies scaling functions φ_ν $(-1 < \nu < 0)$ supported on $[0, 3]$ with samples taken at $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\} + 3\mathbb{Z}$. The QMF H associated to $\varphi = \varphi_\nu$ has coefficients $h_0 = \frac{\nu(\nu-1)}{2(1+\nu^2)}$, $h_1 = \frac{1-\nu}{2(1+\nu^2)}$, $h_2 = \frac{\nu+1}{2(1+\nu^2)}$, $h_3 = \frac{\nu(\nu+1)}{2(1+\nu^2)}$, and $\varphi(1) = \frac{\nu-1}{2\nu}$, $\varphi(2) = \frac{1+\nu}{2\nu}$. We use the dilation equation (4) twice to compute the values of φ at the points $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ then at $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}$. We then have

$$M^\varphi = \begin{pmatrix} \varphi(1/4) & \varphi(1/2) & \varphi(3/4) \\ \varphi(5/4) & \varphi(3/2) & \varphi(7/4) \\ \varphi(9/4) & \varphi(5/2) & \varphi(11/4) \end{pmatrix}$$

and one computes $\det(M^\varphi) = \frac{(1-\nu)^3}{4(1+\nu^2)^4}$ which is non-zero since $-1 < \nu < 0$.

Beyond this kind of calculation, little is known about the validity of SS1. However, it is simple to write down an interesting condition that is equivalent to the singularity of M^φ in the case of compactly supported scaling functions. Suppose φ is supported on $[0, M]$. Let $M \le 2^J$, choose integers $0 \le l_0 < l_1 < \cdots < l_{M-1} \le 2^J - 1$ and take samples at $\{\frac{l_j}{2^J}\}_{j=0}^{M-1} + M\mathbb{Z}$. Now M^φ is singular if and only if its transpose is singular, in which case there exists $\mathbf{d} = (d_0, d_1, \ldots, d_{M-1}) \in \mathbb{C}^M$ such that $(M^\varphi)^t\mathbf{d} = 0$, that is, $\sum_{j=0}^{M-1} \varphi(\frac{l_j}{2^J} + k)d_j = 0$ for all k. Taking the z-transform of both sides of this equation gives

$$0 = \sum_{j=0}^{M-1} d_j Z_{\mathbb{C}}\varphi(\frac{l_j}{2^J}, z) = \sum_{j=0}^{M-1} d_j T_H^J M^{l_j}\Phi_{\mathbb{C}}(z) = T_H^J(\sum_{j=0}^{M-1} d_j M^{l_j}\Phi_{\mathbb{C}})(z) = T_H^J(D\Phi_{\mathbb{C}})(z)$$

where $D(z) = \sum_{j=0}^{M-1} d_j z^{-l_j}$ is a backwards trigonometric polynomial of degree no greater than $2^J - 1$. Hence M^φ is singular if and only if there exists $D(z) = \sum_{j=0}^{M-1} d_j z^{-l_j}$, $0 \le l_j \le 2^J - 1$ with $T^J(D\Phi_{\mathbb{C}}) = 0$.

SS2: This scheme is valid whenever the matrix N^φ is invertible. When the sampling points are known and the scaling function values $\varphi(t_l - k)$ are computable, N^φ may be realized explicitly and its invertibility (or otherwise) may be checked directly. As an example of this procedure, suppose φ is supported on $[0, 3]$ and the sampling points are $t_0 = \frac{1}{2}$, $t_1 = 1$, $t_2 = \frac{3}{2}$, $t_3 = 2$, $t_4 = \frac{5}{2}$. Then N^φ takes the form

$$N^\varphi = \begin{pmatrix} 0 & 0 & \varphi(\frac{1}{2}) & \varphi(\frac{3}{2}) & \varphi(\frac{5}{2}) \\ 0 & 0 & \varphi(1) & \varphi(2) & 0 \\ 0 & \varphi(\frac{1}{2}) & \varphi(\frac{3}{2}) & \varphi(\frac{5}{2}) & 0 \\ 0 & \varphi(1) & \varphi(2) & 0 & 0 \\ \varphi(\frac{1}{2}) & \varphi(\frac{3}{2}) & \varphi(\frac{5}{2}) & 0 & 0 \end{pmatrix}.$$

When φ is the quadratic B-spline, $\det(N^\varphi)$ is easily shown to be 8^{-3}. When $\varphi = \varphi_\nu$ ($-1 < \nu < 0$) is a Daubechies scaling function supported on $[0, 3]$, a simple calculation shows that $\det(N^\varphi) = -\frac{(\nu^4+1)^4}{16\nu^3(\nu^2+1)^3} \ne 0$ for ν in the range $-1 < \nu < 0$. Notice that if the t_l's are close together, N^φ becomes ill-conditioned.

Beyond this kind of calculation, little is known about the validity of SS2. To the knowledge of the authors, the only general result on this subject is the following.

Theorem 2 *Let φ be a continuous, orthogonal scaling function supported on $[0, M]$ with $\varphi(1) \ne 0 \ne \varphi(M-1)$. Suppose $\Phi_{\mathbb{C}}(z) = \sum_k \varphi(k)z^k$ has the property that if $\Phi_{\mathbb{C}}(z_0) = 0$ then $\Phi_{\mathbb{C}}(z_0^2) \ne 0$. Let $t_l = \frac{l}{2}$ $(1 \le l \le 2M-1)$ be sampling nodes and N^φ be the $(2M-1) \times (2M-1)$ matrix defined by $(N^\varphi)_{lk} = \varphi(t_l - k)$ $(1 - M \le k \le M - 1$, $1 \le l \le 2M - 1)$. Then N^φ is invertible.*

The conditions of the theorem are certainly satisfied by a wide class of scaling functions including the B-splines and the Daubechies φ_ν class. However, the restriction on the sampling points is quite strong.

Proof. To simplify the notation, in this proof we write $H(z) = H_{\mathbb{C}}(z)$ and $\Phi(z) = \Phi_{\mathbb{C}}(z)$. Given a Laurent polynomial F, we denote by $[F]$ the collection of those integers j for which $\hat{F}(j)$ is non-zero, where $\hat{F}(j)$ is the j^{th} Fourier coefficient of F, i.e., if $F(z) = \sum_{j=l}^m c_j z^k$, $[F] \subset \{l, \ldots, m\}$. Suppose N^φ is singular. Then there exists $\mathbf{d} = (d_1, d_2, \ldots d_{2M-1}) \in \mathbb{C}^{2M-1}$ such that $N^\varphi\mathbf{d} = 0$, that is,

$$0 = \sum_{k=1}^{2M-1} (N^\varphi)_{lk} d_k = \sum_{k=1}^{2M-1} \varphi\left(\frac{k}{2} + l\right) d_k \qquad (23)$$

for all l. Taking the z-transform of both sides of (23) gives $T(D\Phi) = 0$ where $D(z) = \sum_{k=1}^{2M-1} d_k z^{-k}$. Hence $D(z)\Phi(z) = \lambda(z)H(-z)$ for some odd Laurent polynomial λ. Now $[H(-z)\lambda(z)] = [D(z)\Phi(z)] \subset \{2 - 2M, \ldots, M-2\}$ while $[H(-z)] = \{0, \ldots, M\}$, so $[\lambda(z)] \subset \{2-2M, \ldots, -2\}$. But λ is an odd polynomial, so $[\lambda(z)] \subset \{3-2M, \ldots, -3\}$ and $[D(z)\Phi(z)] = [H(-z)\lambda(z)] \subset \{3-2M, \ldots, M-3\}$. Since $\phi(1) \ne 0 \ne \varphi(M-1)$,

$[\Phi(z)] = \{1, \ldots, M-1\}$ and we conclude that $[D(z)] \subset \{2-2M, \ldots, -2\}$, that is, the top and bottom coefficients of D are zero. Then no negative powers of z appear in the identity

$$z^{2M-3} D(z)\Phi(z) = z^{2M-3}\lambda(z) H(-z).$$

We claim that $\Phi(z)$ divides $z^{2M-3}\lambda(z)$. It is enough to show that $\Phi(z)$ and $H(-z)$ have no common factors. To see this, note that if $\Phi(z_0) = H(-z_0) = 0$, then $\Phi(z_0^2) = H(z_0)\Phi(z_0) + H(-z_0)\Phi(-z_0) = 0$, contradicting the condition on the zeroes of Φ. Since λ is odd, we can write

$$z^{2M-3}\lambda(z) = \Phi(z)\Phi(-z)E(z) \tag{24}$$

with E an even polynomial. However, the left-hand side has degree at most $2M-3+(-3) = 2M-6$, while the right-hand side has degree at least $M-1+M-1 = 2M-2$. Hence, we've reached a contradiction and conclude that N^φ is invertible. ■

4.3 Validation of Oversampling Techniques

We aim first to demonstrate the role of continuity, orthogonality and compact support in the context of sampling.

Theorem 3 *Let φ be a compactly supported function in $L^2(\mathbb{R})$ which is Lipschitz continuous of order $\alpha > 0$ and orthogonal to its integer shifts. Let $L \geq 2$ be an integer, $0 \leq t_0 < t_1 < \cdots < t_{L-1} < 1$, $\mathbf{t} = (t_0, t_1, \ldots, t_{L-1})$ and $\Delta(\mathbf{t}) = \max_{1\leq l\leq L-1}\{t_0, t_l - t_{l-1}, 1 - t_{L-1}\}$. Then*

$$\sum_{l=0}^{L-1} |Z_{\mathbb{C}}\varphi(t_l, z)|^2 \geq c > 0 \tag{25}$$

for all $z \in \mathbb{C}$ provided $\Delta(\mathbf{t})$ is sufficiently small.

By restricting z to lie on the unit circle, (25) becomes (11) and we have a criterion for the validity of OS1. However, (25) may also be viewed as equivalent to the co-primality of the polynomials $\{Z_{\mathbb{C}}\varphi(t_l, z)\}_{l=0}^{L-1}$, so we also have a criterion for the validity of OS2.

Corollary 4 *Let φ be as in Theorem 3. Then oversampling schemes OS1 and OS2 are valid in $V(\varphi)$ with oversampling rate N.*

Proof. To prove Theorem 3, first note that the orthogonality of the integer shifts of φ may be recast in the Fourier domain as $\sum_k |\hat{\varphi}(\xi+k)|^2 \equiv 1$, in the Zak domain as $\int_0^1 |Z\varphi(x,\xi)|^2\,dx \equiv 1$, and in the complexified Zak domain as $\int_0^1 Z_{\mathbb{C}}\varphi(x,z)\overline{Z_{\mathbb{C}}\varphi(x,z^*)}\,dx \equiv 1$.

Suppose φ is supported on $[0, M]$. Without loss of generality, we may also assume that $\int_0^1 |\varphi(x)|^2\,dx > 0$. Hence, there exists $\delta_1 > 0$ such that if $\Delta(\mathbf{t}) < \delta_1$ then, with $t_{-1} = 0$ and $t_L = 1$,

$$\sum_{l=0}^{L}(t_l - t_{l-1})|\varphi(t_l)|^2 = \sum_{l=0}^{L}(t_l - t_{l-1})|Z_{\mathbb{C}}\varphi(t_l, 0)|^2 > 0.$$

By the continuity of $Z_{\mathbb{C}}\varphi$, there exists $r > 0$ such that $\sum_{l=0}^{L} |Z_{\mathbb{C}}\varphi(t_l, z)|^2 > 0$ if $|z| < r$. Also, since the zeroes of $Z_{\mathbb{C}}\varphi(t, \cdot)$ are continuous functions of t, and $Z_{\mathbb{C}}\varphi(x+1, z) = z^{-1}Z_{\mathbb{C}}\varphi(x, z)$, there exists $R > 0$ such that $Z_{\mathbb{C}}\varphi(t, z)$ never vanishes outside $|z| < R$. Hence we have

$$\sum_{l=0}^{L}(t_l - t_{l-1})|Z_{\mathbb{C}}\varphi(t_l, z)|^2 > 0$$

for $|z| < r$ and $|z| > R$. We may also assume, without loss of generality, that $r < 1/2$ and $R > 2$.

Suppose now that $Z_{\mathbb{C}}\varphi(t_l, z_0) = 0$ for $0 \le l \le L-1$ and some z_0 in the annulus $r \le |z| \le R$. For each $z \in \mathbb{C}$, let

$$A_\varphi(z) = \sum_{l=0}^{L} \int_{t_{l-1}}^{t_l} [Z_{\mathbb{C}}\varphi(t,z)\overline{Z_{\mathbb{C}}\varphi(t,z^*)} - Z_{\mathbb{C}}\varphi(t_l,z)\overline{Z_{\mathbb{C}}\varphi(t_l,z^*)}]\, dt.$$

Then, by the orthogonality condition, we have $A(z_0) = 1$. However, for arbitrary z, $A_\varphi(z)$ may be estimated by

$$\begin{aligned} |A_\varphi(z)| &\le \sum_{l=0}^{L} \int_{t_{l-1}}^{t_l} |Z_{\mathbb{C}}\varphi(t,z)||Z_{\mathbb{C}}\varphi(t,z^*) - Z_{\mathbb{C}}\varphi(t_l,z^*)|\, dt \\ &+ \sum_{l=0}^{L} \int_{t_{l-1}}^{t_l} |Z_{\mathbb{C}}\varphi(t,z) - Z_{\mathbb{C}}\varphi(t_l,z)||Z_{\mathbb{C}}\varphi(t_l,z^*)|\, dt = I + II. \end{aligned}$$

But for $r \le |z| \le R$,

$$\begin{aligned} I &\le \sum_{l=0}^{L} \int_{t_{l-1}}^{t_l} \sum_{k=0}^{M-1} |\varphi(t+k)||z|^k \sum_{j=0}^{M-1} |\varphi(t+j) - \varphi(t_l+j)||z|^{-j}\, dt \\ &\le c_\varphi \|\varphi\|_\infty \sum_{l=0}^{L} |z|^k \sum_{k=0}^{M-1} \sum_{j=0}^{M-1} R^k r^{-j} \int_{t_{l-1}}^{t_l} |t - t_l|^\alpha\, dt \\ &\le \frac{2c_\varphi \|\varphi\|_\infty (R^M - 1)(r^{-M} - 1)\Delta(\mathbf{t})^{\alpha+1}}{\alpha+1} \end{aligned}$$

where c_φ is the Lipschitz constant associated with φ, that is, $c_\varphi = \sup_{|x-y|>0} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha}$. An identical estimate may be made for integral II and we conclude that

$$|A_\varphi(z)| \le \frac{4c_\varphi \|\varphi\|_\infty (R^M - 1)(r^{-M} - 1)\Delta(\mathbf{t})^{\alpha+1}}{\alpha+1}$$

for all z in $r \le |z| \le R$. However, $A_\varphi(z_0) = 1$, thus contradicting (4.3) whenever

$$\Delta(\mathbf{t}) < \delta_2 = \left(\frac{\alpha+1}{4c_\varphi \|\varphi\|_\infty (R^M - 1)(r^{-M} - 1)} \right)^{1/(\alpha+1)}.$$

Hence, with $\Delta(\mathbf{t}) < \min\{\delta_1, \delta_2\}$, the theorem is proved. ■

Note that in the case of uniform sampling ($t_l = \frac{l+1}{L}$), $\Delta(\mathbf{t}) = \frac{1}{L}$ and the bound on $\Delta(\mathbf{t})$ may be written $L > \max\{\delta_1^{-1}, \delta_2^{-1}\}$.

Although Theorem 3 tells us that schemes OS1 and OS2 are valid reconstruction techniques when the sampling rate is sufficiently high, the estimate of the appropriate rate is not very helpful as it depends on quantities such as Lipschitz constants which are difficult to compute. We aim to provide conditions on φ that validate our oversampling schemes with sampling rates that depend only on the length of the support of φ or its Fourier transform. As an example of the latter type of estimate we present the following theorem, where the effect of bandlimitedness of the generator is considered. In this case we can provide an estimate of an appropriate (uniform) sampling rate L for which (11) is valid and the form (12) of the sampling functions simplifies dramatically.

Theorem 5 *Suppose φ is bandlimited to $[0, M]$ and the integer translates of φ are orthonormal. Then equation (11) becomes $\sum_{l=0}^{M-1} |Z\varphi(\frac{l}{M} + y, \xi)|^2 = M$ for all ξ.*

Corollary 6 *If φ is as in Theorem 5, then the oversampling scheme OS1 is valid in $V(\varphi)$, i.e., each $f \in V(\varphi)$ admits a sampling representation of the form*

$$f(t) = \sum_{m=0}^{M-1} \sum_l f\left(y + \frac{m}{M} + l\right) S_m^y(t-l)$$

with $S_m^y(t) = \frac{1}{M}\sum_k \overline{\varphi(\frac{m}{M} + y + k)}\varphi(t+k) \in V(\varphi)$.

Here the sampling rate of M is the same as that prescribed by the Classical Sampling Theorem for signals bandlimited to $[0, M]$.

Proof. To prove Theorem 5, note that by the Poisson summation formula and the orthogonality condition on φ,

$$\begin{aligned}
\frac{1}{M}\sum_{l=0}^{M-1}\left|Z\varphi\left(\frac{l}{M}+y,\xi\right)\right|^2 &= \frac{1}{M}\sum_{l=0}^{M-1}\left|Z\hat{\varphi}\left(-\xi,\frac{l}{M}+y\right)\right|^2 \\
&= \frac{1}{M}\sum_{j=0}^{M-1}\sum_{n=0}^{M-1}\hat{\varphi}(-\xi+j)\overline{\hat{\varphi}(-\xi+n)}\sum_{l=0}^{M-1} e^{2\pi i(j-n)(\frac{l}{M}+y)} \\
&= \sum_{j=0}^{M-1}|\hat{\varphi}(-\xi+j)|^2 \equiv 1.
\end{aligned}$$

This proves the theorem. ■

Proof. To prove Corollary 6, note that since $\sum_{n=0}^{M-1}|Z\varphi(\frac{n}{M}+y,\xi)|^2 = M$, from equation (12) we have

$$\begin{aligned}
S_m^y(t) &= \frac{1}{M}\int_0^1 Z\varphi(t,\xi)\overline{Z\varphi\left(\frac{m}{M}+y,\xi\right)}\,d\xi \\
&= \frac{1}{M}\sum_{k=0}^{M-1}\int_0^1 e^{2\pi i k\xi}\overline{Z\varphi\left(\frac{m}{M}+y,\xi\right)}\,d\xi\,\varphi(t+k) = \frac{1}{M}\sum_{k=0}^{M-1}\overline{\varphi\left(\frac{m}{M}+y+k\right)}\varphi(t+k)
\end{aligned}$$

where the last equality comes from an application of the quasiperiodicty relation and the Zak inversion formula. ■

To retrieve the cardinal sine functions of the Classical Sampling Theorem, put $\hat{\varphi} = \chi_{[0,1]}$ so that $\varphi(t) = e^{\pi i t}\,\mathrm{sinc}\,t$. Then since $|Z\varphi(t,\xi)| = |e^{-2\pi i[t]\xi}| = 1$, there is only one sampling function, namely

$$S^y(t) = \sum_k \overline{\varphi(y+k)}\varphi(t+k) = \frac{e^{\pi i(t-y)}}{\pi^2}\sum_k \frac{\sin(\pi y)\sin(\pi t)}{(y+k)(t+k)} = e^{\pi i(t-y)}\frac{\sin\pi(t-y)}{\pi(t-y)} = \varphi(t-y).$$

Now we turn to the compactly supported case. Theorem 3 says that orthogonality, compact support and smoothness of φ suffices to validate OS1 with a sufficiently high sampling rate. If we insist that φ be a scaling function, we can validate oversampling at a more effective rate, as the next two results will show. In fact, the sampling rate is controlled in terms of zeroes of the QMF H.

Theorem 7 *Suppose that φ is a continuous scaling function supported on $[0, 2^J+1]$, and the QMF $H(\xi)$ associated with φ has no zeroes except at $\xi = \frac{1}{2}$. Then for all ξ,*

$$\sum_{l=0}^{2^J-1}|Z\varphi(\frac{l}{2^J},\xi)|^2 \geq c > 0.$$

Proof. Notice that

$$Z\varphi\left(\frac{l}{2^J},\xi\right) = T^J M^l \Phi(\xi) \quad = \quad \sum_{j=0}^{2^J-1} e^{-2\pi i j l/2^J}\,\Phi\left(\frac{\xi+j}{2^J}\right)\prod_{p=1}^{J} H\left(\frac{\xi+j}{2^p}\right)$$
$$= \quad 2^{J/2}\mathcal{F}_{2^J}\left(\Phi\left(\frac{\xi+\cdot}{2^J}\right)\prod_{p=1}^{J} H\left(\frac{\xi+\cdot}{2^p}\right)\right)(l)$$

where $\mathcal{F}_{2^J}$ is the 2^J-point (fast) Fourier transform. Therefore, if $Z\varphi(\frac{l}{2^J},\xi_*) = 0$ for all l then by Fourier uniqueness,

$$\Phi\left(\frac{\xi_*+k}{2^J}\right)\prod_{p=1}^{J} H\left(\frac{\xi_*+k}{2^p}\right) = \Phi\left(\frac{\xi_*+k}{2^J}\right)\hat{\varphi}_J(-\xi_*-k) = 0 \text{ for } 1-2^{J-1} \le k \le 2^{J-1}. \tag{26}$$

Since φ is supported on $[0, 2^J+1]$, $\Phi_{\mathbb{C}}$ has degree 2^J and has a zero at $z = 0$. Hence $\Phi_{\mathbb{C}}$ can have at most $2^J - 1$ zeroes on the unit circle. Consequently, (26) implies that $\prod_{p=1}^{J} H(\frac{\xi_*+k}{2^p}) = 0$ for at least 1 value of k (say $k = \kappa$). Since H has zeroes on the unit circle at $\xi = \frac{1}{2}$ only, we have $\frac{\xi_*+\kappa}{2^p} = Q + \frac{1}{2}$ for some integer Q and $1 \le p \le J$. Hence $\xi_* = 2^p(Q+\frac{1}{2})-\kappa = 2^pQ+2^{p-1}-\kappa \in \mathbb{Z}$. But this means that $\Phi(2^pQ+2^{p-1}-\kappa) = 0$, a contradiction since $\Phi(m) = 1$ for all integers m. ∎

Corollary 8 *If φ is supported on $[0, 2^J+1]$, is continuous and satisfies a dilation equation with filter $H(\xi)$ having no zeroes other than at $\xi = \frac{1}{2}$, then oversampling scheme OS1 is valid in $V(\varphi)$ with sampling rate 2^J.*

As an example, the Daubechies scaling functions φ_ν are supported on $[0, 3]$ and satisfy the condition of the theorem with $J = 1$, hence sampling at half-integers gives an effective oversampling scheme.

While many commonly utilized QMF's, including the Haar QMF $H(\xi) = e^{\pi i\xi}\cos\pi\xi$, satisfy the condition of Theorem 7, others have *ripples* near 0 and $\frac{1}{2}$ which enable them to have zeroes at points on the circle other than $\xi = \frac{1}{2}$ [4, p 200]. The following curious result allows H to have other unimodular zeroes, though at the expense of a higher sampling rate.

Theorem 9 *Let φ be a continuous orthogonal scaling function supported on $[0, M]$ with associated QMF $H(\xi)$. Suppose $H(\xi) \ne 0$ for $|\xi| \le A$ ($A > \frac{1}{4}$) and let J be the least integer for which $2^J \ge \frac{2}{4A-1}$. Suppose $Z\varphi(\frac{l}{2^Q},\xi_*)$ for $0 \le l \le 2^Q - 1$ and some ξ_* with $2^Q \ge 2\max\{A\|\Phi'\|_\infty, 2^J\}$. Then $\varphi \equiv 0$.*

Corollary 10 *Let φ, H, Q be as in the statement of Theorem 9. Then uniform oversampling scheme OS1 is valid in $V(\varphi)$ at the rate 2^Q.*

The appearance of $\|\Phi'\|_\infty$ in Theorem 9 and Corollary 10 is unsatisfying since this quantity can be difficult to compute. Moreover, it is not possible to estimate $\|\Phi'\|_\infty$ solely in terms of the length M of the support of φ. For example, the Daubechies scaling functions $\varphi = \varphi_\nu$ supported in $[0, 3]$ satisfy $\|\Phi'_\nu\|_\infty = 2\pi(\frac{\nu+1}{2\nu}) \to \infty$ as $\nu \to 0$. It is possible to replace $\|\Phi'\|_\infty$ in the theorem and corollary by $\pi\sqrt{M}(M+1)\|\Phi\|_2/\sqrt{3}$, but again $\|\Phi\|_2$ cannot be estimated by a function of M. Strengthening the condition on zeroes of H to $H(\xi) \ne 0$ for $|\xi| < B$ with $B > 2^{-J}$, $J \ge 3$ gives improved constants in the estimates, hence better sampling rates. Proofs of these claims are left to the reader.

Proof. To prove Theorem 9, note that since $Z\varphi(\frac{l}{2^Q},\xi_*) = 0$ for all l, we trivially have $Z\varphi(\frac{l}{2^J},\xi_*) = 0$ for all l. Since the integer shifts of φ_J (as defined in section 4.1) are orthogonal, we have $\sum_k |\hat{\varphi}_J(-\xi_*-k)|^2 = 1$ from which we see that $\hat{\varphi}(-\xi_*-k) \ne 0$ for at least one value of k, say $k = \kappa$ ($1-2^{J-1} \le \kappa \le 2^{J-1}$). By (26), $\Phi(\frac{\xi_*+\kappa}{2^J}) = 0$. If $\hat{\varphi}_J^{per}(\xi) = \sum_l \hat{\varphi}_J(\xi+2^Jl)$ is the 2^J-periodisation of $\hat{\varphi}_J$, then

$$\hat{\varphi}_{J+1}(-\xi_*-k) = H\left(\frac{-\xi_*-k}{2^{J+1}}\right)\hat{\varphi}_J^{per}(-\xi_*-k)\chi_{[-2^J,2^J]}(\xi_*+k). \tag{27}$$

But $|\frac{\xi_*+\kappa}{2^{J+1}}| \leq \frac{2^{J-1}+1}{2^{J+1}} = \frac{1}{4} + 2^{-J-1} < A$, so $H(\frac{\xi_*+\kappa}{2^{J+1}}) \neq 0$. Since $\hat{\varphi}_J(-\xi_* - k) \neq 0$, (27) implies that $\hat{\varphi}_{J+1}(-\xi_* - \kappa) \neq 0$. On the other hand, by (26), $\hat{\varphi}_{J+1}(-\xi_* - \kappa)\Phi(\frac{\xi_*+\kappa}{2^{J+1}}) = 0$, so $\Phi(\frac{\xi_*+\kappa}{2^{J+1}}) = 0$. We have therefore found a zero $\xi_{J+1} = \frac{\xi_*+\kappa}{2^{J+1}}$ of Φ with $|\xi_{J+1}| \leq \frac{2^{J_1}+1}{2^{J+1}} = \frac{1}{4} + 2^{-J-1} < A$. We also have

$$\hat{\varphi}_{J+2}(-\xi_* - \kappa) = H\left(\frac{\xi_* + \kappa}{2^{J+2}}\right) H\left(\frac{\xi_* + \kappa}{2^{J+1}}\right) \hat{\varphi}_J(-\xi_* - \kappa)\chi_{[-2^{J+1},2^{J+1}]}(\xi_* + \kappa).$$

But $\hat{\varphi}_J(-\xi_* - \kappa) \neq 0$, and since $|\frac{\xi_*+\kappa}{2^{J+2}}| < |\frac{\xi_*+\kappa}{2^{J+1}}| \leq A$, we have $H(\frac{\xi_*+\kappa}{2^{J+2}})H(\frac{\xi_*+\kappa}{2^{J+1}}) \neq 0$ and we conclude that $\hat{\varphi}_{J+2}(-\xi_* - \kappa) \neq 0$. By (26), $\Phi(\frac{\xi_*+\kappa}{2^{J+2}}) = 0$ and we have found a zero $\xi_{J+2} = \frac{\xi_*+\kappa}{2^{J+2}}$ of Φ with $|\xi_{J+2}| < A/2$. Continuing in this way, we find a zero $\xi_Q = \frac{\xi_*+\kappa}{2^Q}$ of Φ with $|\xi_Q| < 2^{1-Q}A$. However, Φ is non-zero on a neighbourhood of the origin. In fact, if $\Phi(\eta) = 0$, then $1 = |\Phi(\eta) - 1| \leq |\eta|\|\Phi'\|_\infty$. Consequently, every zero η of Φ satisfies

$$|\eta| \geq \|\Phi'\|_\infty^{-1}. \tag{28}$$

But $\Phi(\xi_Q) = 0$, thus contradicting (28) provided $2^Q \geq 2A\|\Phi'\|_\infty$. ■

As discussed in Section 3, the oversampling procedures that these theorems provide do not generate compactly supported sampling functions – despite the fact that the associated scaling functions are compactly supported. To compute compactly supported sampling functions, we need more precise information about the zeroes of $Z_{\mathbb{C}}\varphi$.

Theorem 11 *Suppose that φ is a continuous scaling function for an MRA supported on $[0, M]$. Then the polynomials $\{Z_{\mathbb{C}}\varphi(\frac{l}{2^{M-2}}, z)\}_{l=0}^{2^{M-2}-1}$ have no common zeroes.*

Corollary 12 *If φ is as in Theorem 11, then the uniform oversampling scheme OS2 is valid with oversampling rate 2^{L-2}, and the sampling functions have compact support.*

To prove Theorem 11, we need several preliminary results on the zeroes of the Zak transforms of scaling functions.

Lemma 13 *Suppose φ is a continuous scaling function supported on $[0, \infty)$ such that $\varphi(\frac{k}{2^J}) = 0$ for $0 \leq k \leq 2^J - 1$. Then*

$$\varphi\left(2^j + \frac{k2^j}{2^{J-1}}\right) = 0 \textit{ for } 0 \leq k \leq 2^{J-1} - 1,\ 0 \leq j \leq J - 1. \tag{29}$$

The proof of 13 requires a double induction (on k and J). The proof is omitted, but the idea of the proof may be easily transmitted with the following example. Suppose φ is as in the statement of the lemma and $\varphi(\frac{k}{8}) = 0$ for $0 \leq k \leq 7$, i.e., $J = 3$. By the dilation equation (4) we see that by putting $t = 1$ and using the assumption that $\varphi(\frac{1}{2}) = 0$, we have

$$0 = \frac{1}{2}\varphi\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty} h_k\varphi(1 - k) = h_0\varphi(1),$$

from which we obtain $\varphi(1) = 0$ (since, without loss of generality, we may assume $h_0 \neq 0$). Using the dilation equation successively at $t = \frac{5}{4}, \frac{3}{2}, \frac{7}{4}$ gives $\varphi(\frac{5}{4}) = \varphi(\frac{3}{4}) = \varphi(\frac{7}{4}) = 0$. Now putting $t = \frac{5}{2}, 3, \frac{7}{2}, 4$ gives $\varphi(\frac{5}{2}) = \varphi(3) = \varphi(\frac{7}{2}) = \varphi(4) = 0$. Putting $t = 5, 6, 7, 8$ gives $\varphi(5) = \varphi(6) = \varphi(7) = \varphi(8) = 0$.

Corollary 14 *If φ is a continuous scaling function supported on $[0, 2^J+2]$ and $\varphi(\frac{k}{2^J}) = 0$ for $0 \leq k \leq 2^J-1$, then $\varphi \equiv 0$.*

Proof. By the previous lemma, $\varphi(2^j + \frac{k2^j}{2^{J-1}}) = 0$ for $0 \leq k \leq 2^J - 1$ and $0 \leq j \leq J - 1$. In particular, when $k = m2^{J-1-j}$ with $0 \leq m \leq 2^j - 1$ we have $\varphi(2^j + m) = 0$ for $0 \leq j \leq J - 1$ and $0 \leq m \leq 2^j - 1$, i.e., $\varphi(l) = 0$ for all $0 \leq l \leq 2^{J-1} + 2^{J-1} - 1 = 2^J - 1$. A further application of the dilation equation yields

$\varphi(2^J) = 0$. Since $\varphi(l) = 0$ for $l \geq 2^J + 2$ and $\sum_k \varphi(k) = 1$, we must have $\varphi(2^J + 1) = 1$, i.e., $\varphi(l) = \delta_{2^J+1-l}$. So φ is a continuous compactly supported cardinal scaling function. By the result of Xia and Zhang [17] which rules out this behaviour, we have $\varphi \equiv 0$. ∎

Simply by noting that $Z_{\mathbb{C}}\varphi(x, 0) = \varphi(x)$, we have the following result.

Corollary 15 *If φ is a continuous scaling function supported on $[0, 2^J + 2]$ and $Z_{\mathbb{C}}\varphi(\frac{k}{2^J}, 0) = 0$ for $0 \leq k \leq 2^J - 1$, then $\varphi \equiv 0$.*

Since $2^l > l$ for all positive integers l, we have the following simple consequence of Corollary 15:

Corollary 16 *If φ is a continuous scaling function supported on $[0, M]$ and $Z_{\mathbb{C}}\varphi(\frac{k}{2^{M-2}}, 0) = 0$ for $0 \leq k \leq 2^{M-2} - 1$, then $\varphi \equiv 0$.*

Now we can prove Theorem 11.

Proof. To prove Theorem 11, suppose z_0 is a common zero for $\{Z_{\mathbb{C}}\varphi(\frac{l}{2^{M-2}}, z)\}_{l=0}^{2^{M-2}-1}$. Since $Z_{\mathbb{C}}\varphi(x, 1) = 1$ for all x we see that $z_0 \neq 1$. Further, by Corollary 16, $z_0 \neq 0$. Suppose then that $z_1^2 = z_0$. Then for each $0 \leq l \leq 2^{M-2} - 1$, we have

$$\begin{aligned} 0 &= Z_{\mathbb{C}}\varphi\left(\frac{l}{2^{M-2}}, z_0\right) = H(z_1)Z_{\mathbb{C}}\varphi\left(\frac{l}{2^{M-3}}, z_1\right) + H(-z_1)Z_{\mathbb{C}}\varphi\left(\frac{l}{2^{M-3}}, -z_1\right), \text{ and} \\ 0 &= Z_{\mathbb{C}}\varphi\left(\frac{l + 2^{M-3}}{2^{M-2}}, z_0\right) = \frac{1}{z_1}\left[H(z_1)Z_{\mathbb{C}}\varphi\left(\frac{l}{2^{M-3}}, z_1\right) - H(-z_1)Z_{\mathbb{C}}\varphi\left(\frac{l}{2^{M-3}}, -z_1\right)\right]. \end{aligned}$$

Hence $H(z_1)Z_{\mathbb{C}}\varphi(\frac{l}{2^{M-3}}, z_1) = 0 = H(-z_1)Z_{\mathbb{C}}\varphi(\frac{l}{2^{M-3}}, -z_1)$. However, by the QMF condition (18), $H(z_1)$ and $H(-z_1)$ cannot both be zero. By swapping z_1 and $-z_1$ if necessary, we have $Z_{\mathbb{C}}\varphi(\frac{l}{2^{M-3}}, z_1) = 0$ for $0 \leq l \leq 2^{M-3} - 1$. In particular, since $z_1 \neq z_0$, putting $l = 0$ we see that we have another zero of $\Phi(z) = Z_{\mathbb{C}}\varphi(0, z)$. We say z_0 is a common zero at level 0 and z_1 is a common zero at level 1 and formalize this by starting a tree of zeroes as in Figure 1.

We can continue this process, producing common zeroes z_2 at level 2, z_3 at level 3 and so on until finally we reach z_{M-2} at level $M - 2$. If $z_0 \notin \mathbb{T}$, then $0, z_0, z_1, \ldots, z_{M-2}$ are M distinct zeroes of Φ, a trigonometric polynomial of degree $M - 1$. Hence we have a contradiction.

We are left with the situation where $z_0 \in \mathbb{T}$. In this case, either the string of zeroes $z_0, z_1, \ldots z_{M-2}$ are distinct, in which case we have our contradiction and are finished, or the string contains a repetition. Suppose $z_j = z_k$ for some $j < k$. Then $z_0 = z_j^{2^j} = z_k^{2^j} = z_{k-j}$, and we see that the first repetition occurs between z_0 and some z_m, $m \geq 2$ ($z_1 \neq z_0$ since $z_1^2 = z_0$ and $z_0 \neq 1$). We may assume, therefore, that $z_0, z_1, \ldots z_{m-1}$ are all distinct with $m \geq 2$. We claim that the string $z_0, z_1, \ldots, z_{m-1}$ must branch at some point, i.e., for some $0 \leq j \leq m - 1$, $w_j = -z_j$ is another zero at level j. To see this, suppose the contrary, i.e., for all $0 \leq j \leq m - 1$, z_j is a zero at level j but $w_j = -z_j$ is not. Since $Z_{\mathbb{C}}\varphi(\frac{l}{2^{M-2-j}}, z_j) = Z_{\mathbb{C}}\varphi(\frac{l}{2^{M-2-j}} + \frac{1}{2}, z_j) = 0$ ($0 \leq l \leq 2^{M-2-j} - 1$), we have

$$0 = H(z_{j+1})Z_{\mathbb{C}}\varphi\left(\frac{l}{2^{M-3-j}}, z_{j+1}\right) = H(-z_{j+1})Z_{\mathbb{C}}\varphi\left(\frac{l}{2^{M-3-j}}, -z_{j+1}\right).$$

Since $Z_{\mathbb{C}}\varphi(\frac{l}{2^{M-3-j}}, -z_{j+1}) \neq 0$, we must have $H(-z_{j+1}) = 0$ and, again by (2.4), $|H(z_{j+1})| = 1$. Consequently the string $z_0, z_1, \ldots, z_{m-1}$ is a τ-cycle for which $|H(z_n)| = 1$ for $0 \leq n \leq m - 1$, a contradiction of the τ-cycle condition. We conclude that the string branches at some z_j as in Figure 2.

We can use w_j as the seed for a new string of zeroes $w_j, w_{j+1}, \ldots$ which is easily seen to be distinct from the string $z_0, z_1, \ldots, z_{m-1}$. Either this new string reaches level $M - 2$, in which case we are finished, or there is a repetition in the new string, in which case it branches. Continuing in this way we notice that each new string has length at least two since if $w_{j+2} = w_{j+1}$ then $w_{j+1}^2 = w_{j+2}^2 = w_{j+1}$ which implies that $w_{j+1} = 1$, a contradiction of the fact that $Z_{\mathbb{C}}\varphi(x, 1) = 1$ for all x. The worst possible scenario (in the sense of production of the fewest new zeroes per string) is that in which each new string has length 2. It is clear, however, that

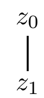

Figure 1: Seed of the tree of zeroes of Φ

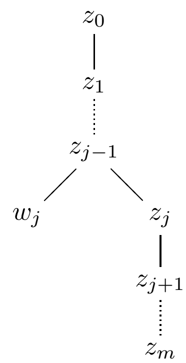

Figure 2: Branched tree of zeroes

after 1 branching we reach level 2. Further, the m^{th} level is reached after no more than $m-1$ branchings, and the $(M-2)^{\text{th}}$ level is reached in no more than $M-3$ branchings. In this way we are assured of finding sufficient zeroes for a contradiction. The proof is complete. ■

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