A NOTE ON HARMONIC MEASURE

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Abstract. Let Ω be a subregion of $\{z : |z| < 1\}$ for which the Dirichlet problem is solvable, assume that $0 \in \Omega$ and let ω_Ω denote harmonic measure on $\partial\Omega$ for evaluation at 0. If E is a Borel subset of $\{z : |z| = 1\}$ and $\omega_\Omega(E) > 0$, then we find a simply connected region G, where $0 \in G \subseteq \{z : |z| < 1\}$, $\partial G \subseteq \Omega \cup E$ and $\omega_G(E) > 0$, such that $U := G \cup \Omega$ has the property that ω_U and ω_Ω are boundedly equivalent on ∂U. We mention consequences of this in function theory.

1. Introduction

Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$, let $\mathbb{T}$ denote the unit circle $\{z : |z| = 1\}$ and let m denote normalized Lebesgue measure on $\mathbb{T}$. Let Ω be a Dirichlet region (i.e., a region for which the Dirichlet problem is solvable) such that $0 \in \Omega \subseteq \mathbb{D}$, and let ω_Ω denote harmonic measure on $\partial\Omega$ for evaluation at 0. If Ω is simply connected, E is a Borel subset of $\mathbb{T}$ and $\omega_\Omega(E) > 0$, then one can find a Jordan region W with rectifiable boundary such that $0 \in W \subseteq \Omega$, $\partial W \subseteq \Omega \cup E$ and $m(E \cap \partial W) > 0$ (cf., [4] and [10], Proposition 6.23). In this paper we expose an analogue of this for Dirichlet subregions of $\mathbb{D}$ in general. First observe the obvious fact that if G is a simply connected region such that $0 \in G \subseteq \Omega$ and $\omega_G(\mathbb{T} \cap \partial G) > 0$, then by the Maximum Principle, $\omega_\Omega(\mathbb{T} \cap \partial G) > 0$ and hence $\omega_\Omega(\mathbb{T}) > 0$. The main result of this paper (Theorem 3.6) shows that a close relative of this condition characterizes when $\omega_\Omega(\mathbb{T}) > 0$. Indeed, we show that if E is a Borel subset of $\mathbb{T}$ and $\omega_\Omega(E) > 0$, then there is a simply connected region G, where $0 \in G \subseteq \mathbb{D}$, $\partial G \subseteq \Omega \cup E$ and $\omega_G(E) > 0$, such that $U := G \cup \Omega$ has the property that ω_U and ω_Ω are boundedly equivalent on ∂U. In essence, the distribution of ω_Ω on the part of the boundary of Ω that lies in G does not effectively diminish the distribution of ω_Ω elsewhere on $\partial\Omega$. This result has consequences in function theory and also is closely related to some recent work on so-called *champagne bubbles* subregions of the disk; cf., Remarks 3.7.

2. Preliminaries

Let A denote two-dimensional Lebesgue measure (i.e., area measure) on $\mathbb{D}$ and, as mentioned earlier, let m denote normalized Lebesgue measure on $\mathbb{T}$. The Poisson kernel on $\mathbb{T}$ for evaluation at some point w in $\mathbb{D}$ is given by

$$P_w(\xi) = \frac{1 - |w|^2}{|\xi - w|^2}.$$

This kernel provides the solution to the Dirichlet problem for $\mathbb{D}$ in that, for any continuous real-valued function h on $\mathbb{T}$,

$$w \mapsto \int_{\mathbb{T}} h(\xi) P_w(\xi) dm(\xi)$$

is the unique harmonic function in $\mathbb{D}$ that has continuous extension to $\overline{\mathbb{D}}$ with boundary values h. If μ is a finite, positive Borel measure with support in $\overline{\mathbb{D}}$, then, for z in $\mathbb{D}$, we let $\psi_\mu(z)$ denote the integral

$$\int_{\overline{\mathbb{D}}} \log \left| \tfrac{w-z}{1-\bar{z}w} \right| d\mu(w) \quad \left(= \int_{\mathbb{D}} \log \left| \tfrac{w-z}{1-\bar{z}w} \right| d\mu(w)\right),$$

which is the negative of the *Green's potential* of $\mu|_{\mathbb{D}}$ at z; cf., [8]. Notice that if ν is a representing measure for evaluation at 0 and ν has support in $\overline{\mathbb{D}}$, then, for z in $\mathbb{D}$,

$$\begin{aligned} \psi_\nu(z) &= \int_{\overline{\mathbb{D}}} \log\left|\tfrac{w-z}{1-\bar{z}w}\right| d\nu(w) \\ &= \int_{\overline{\mathbb{D}}} \log|w-z| d\nu(w) =: p_\nu(z), \end{aligned}$$

which is the negative of the *logarithmic potential* of ν at z. For points z and w in $\mathbb{D}$, let $\rho(z,w) = \left|\frac{w-z}{1-\bar{z}w}\right|$ – the so-called *pseudohyperbolic* distance between z and w. One may verify that ρ is indeed a metric on $\mathbb{D}$. For z in $\mathbb{D}$ and $0 < r < 1$, let $D_r(z) = \{w \in \mathbb{D} : \rho(z,w) < r\}$. Once again, let μ be a finite, positive Borel measure that has support in $\overline{\mathbb{D}}$. For z in $\mathbb{D}$ and $0 < r < 1$, we let $\psi_{\mu,r}(z)$ denote the integral

$$\int_{D_r(z)} \log\left|\tfrac{w-z}{1-\bar{z}w}\right| d\mu(w).$$

In addition, we define $Q_{\mu,r}$ on $\mathbb{D}$ by

$$Q_{\mu,r}(z) = \begin{cases} 0 & \text{if } |z| \le \frac{1}{2} \\ \frac{\psi_{\mu,r}(z)}{\log|z|} & \text{if } \frac{1}{2} < |z| < 1. \end{cases}$$

For ξ in $\mathbb{T}$ and $0 < a < 1$, let $S_a(\xi)$ denote the interior of the the closed convex hull of $\{z : |z| \le a\} \cup \{\xi\}$. We call $S_a(\xi)$ a *Stolz* region based at ξ. Notice that $S_a(\xi)$ forms an angle of $2 \cdot \arcsin(a)$ at ξ.

Lemma 2.1. *Let μ be a finite, positive Borel measure with support in $\overline{\mathbb{D}}$. If $\int_{\mathbb{D}} P_w(\xi)d\mu(w) < \infty$ for some ξ in $\mathbb{T}$, then, for any r, $0 < r < 1$, and any Stolz region $S_a(\xi)$,*

$$\int_{S_a(\xi)} \frac{Q_{\mu,r}(z)}{|\xi - z|^2} dA(z) < \infty,$$

and hence

$$\liminf_{\substack{z\to\xi \\ z\in S_a(\xi)}} \frac{\psi_{\mu,r}(z)}{\log|z|} = 0.$$

Proof. We assume, without loss of generality, that $\xi = 1$. Choose constants a and r, $0 < a, r < 1$, and let $S = S_a(1)$. Let $S^* = \{w :$ there exists z in S such that $\rho(w,z) < r\}$. Notice that both S^* and $S^\# := (S^*)^*$ are contained in a Stolz region based at 1. So, there are positive constants c_1, c_2 and c_3 (depending only on a and r) such that

$$\begin{aligned}
\int_S \frac{Q_{\mu,r}(z)}{|1-z|^2} dA(z) &\leq \int_S \left(\frac{1}{|1-z|^2 \log|z|} \cdot \int_{D_r(z)} \log\left|\tfrac{w-z}{1-\overline{z}w}\right| d\mu(w) \right) dA(z) \\
&= \int_S \left(\int_{S^*} \frac{\log\left|\frac{w-z}{1-\overline{z}w}\right|}{|1-z|^2 \log|z|} \cdot \chi_{D_r(z)}(w) d\mu(w) \right) dA(z) \\
&\leq \int_{S^*} \left(\int_{S^\#} \frac{\log\left|\frac{w-z}{1-\overline{z}w}\right|}{|1-z|^2 \log|z|} \cdot \chi_{D_r(w)}(z) dA(z) \right) d\mu(w) \\
&\leq c_1 \cdot \int_{S^*} \frac{1}{|1-w|^3} \left(\int_{S^\#} \log\left|\tfrac{1-\overline{w}z}{w-z}\right| \cdot \chi_{D_r(w)}(z) dA(z) \right) d\mu(w) \\
&\leq c_2 r^2 \log(1/r) \cdot \int_{S^*} \frac{1}{|1-w|} d\mu(w) \\
&\leq c_3 r^2 \log(1/r) \cdot \int_{\mathbb{D}} P_w(1) d\mu(w),
\end{aligned}$$

which, by our hypothesis, is finite. □

Theorem 2.2. *Let μ be a finite, positive Borel measure with support in $\overline{\mathbb{D}}$ and let ξ be in $\mathbb{T}$. If $\{z_n\}_{n=1}^{\infty}$ is any sequence in $\mathbb{D}$ that converges to ξ, then*

$$\int_{\mathbb{D}} P_w(\xi) d\mu(w) \leq \liminf_{n\to\infty} \frac{\psi_\mu(z_n)}{\log|z_n|}.$$

Furthermore, if $\int_{\mathbb{D}} P_w(\xi)d\mu(w) < \infty$, then any Stolz region $S_a(\xi)$ contains a sequence $\{z_n\}_{n=1}^{\infty}$ such that

a) $z_n \longrightarrow \xi$ *and*

b) $\frac{\psi_\mu(z_n)}{\log|z_n|} \longrightarrow \int_{\mathbb{D}} P_w(\xi)d\mu(w)$,

as $n \to \infty$.

Proof. Again, without loss of generality, we assume that $\xi = 1$. Let $\{z_n\}_{n=1}^{\infty}$ be any sequence in $\mathbb{D}$ such that $z_n \longrightarrow 1$, as $n \to \infty$. If $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are sequences of positive functions on some set B, then we define $u_n(w) \sim v_n(w)$ to mean: $\frac{u_n(w)}{v_n(w)} \longrightarrow 1$ uniformly for w in B, as $n \to \infty$. With this notation in hand, choose s, $0 < s < 1$, and notice that if $|w| \leq s$, then

$$\begin{aligned}
\frac{\log\left|\frac{w-z_n}{1-\overline{z}_n w}\right|}{\log|z_n|} &\sim \frac{1-\left|\frac{w-z_n}{1-\overline{z}_n w}\right|}{1-|z_n|} \\
&\sim \frac{1-\left|\frac{w-z_n}{1-\overline{z}_n w}\right|^2}{2(1-|z_n|)} \\
&= \left(\frac{|1-\overline{z}_n w|^2 - |w-z_n|^2}{|1-\overline{z}_n w|^2}\right) /2(1-|z_n|) \\
&= \frac{(1+|z_n|)}{2} \cdot \left(\frac{1-|w|^2}{|1-\overline{z}_n w|^2}\right) \\
&\sim \frac{1-|w|^2}{|1-\overline{z}_n w|^2} \longrightarrow P_w(1),
\end{aligned}$$

uniformly for $|w| \leq s$, as $n \to \infty$. Applying Fatou's Lemma we have:

$$\int_{\mathbb{D}} P_w(1) d\mu(w) \leq \liminf_{n\to\infty} \frac{\psi_\mu(z_n)}{\log|z_n|},$$

and furthermore,

$$\frac{1}{\log|z_n|} \cdot \int_{\{|w|\leq s\}} \log\left|\tfrac{w-z_n}{1-\overline{w}z_n}\right| d\mu(w) \longrightarrow \int_{\{|w|\leq s\}} P_w(1) d\mu(w), \tag{2.2.1}$$

as $n \to \infty$. Continuing, we let $S_a(1)$ be a Stolz region (based at 1) and choose r, $0 < r < 1$. Assuming that $\int_{\mathbb{D}} P_w(1) d\mu(w) < \infty$, we can apply Lemma 2.1 and find a sequence $\{z_n\}_{n=1}^{\infty}$ contained in $S_a(1)$ such that $z_n \longrightarrow 1$ and

$$\frac{\psi_{\mu,r}(z_n)}{\log|z_n|} \longrightarrow 0, \tag{2.2.2}$$

as $n \to \infty$. Moreover, if $\left|\frac{w-z_n}{1-\overline{z}_n w}\right| \geq r$, then there are positive constants c_1, c_2, and c_3 (independent of w and n) such that

$$\begin{aligned} \frac{\log\left|\frac{w-z_n}{1-\overline{z}_n w}\right|}{\log|z_n|} &\leq c_1 \cdot \left(1 - \left|\frac{w-z_n}{1-\overline{z}_n w}\right|^2\right) / (1-|z_n|) \\ &\leq c_2 \cdot \left(\frac{1-|w|^2}{|1-\overline{z}_n w|^2}\right) \\ &\leq c_3 \cdot P_w(1). \end{aligned} \tag{2.2.3}$$

And since $\int_{\mathbb{D}} P_w(1) d\mu(w) < \infty$, we have:

$$\int_{\{s<|w|<1\}} P_w(1) d\mu(w) \longrightarrow 0,$$

as $s \to 1$. Putting this together with (2.2.1) - (2.2.3), we find that

$$\frac{\psi_\mu(z_n)}{\log|z_n|} \longrightarrow \int_{\mathbb{D}} P_w(1) d\mu(w),$$

as $n \to \infty$, for this sequence $\{z_n\}_{n=1}^{\infty}$ in $S_a(1)$ given by Lemma 2.1. □

Discussion 2.3. Let Ω be a bounded Dirichlet region and let μ be a finite, positive Borel measure with support in $\overline{\Omega}$. If $h \in C_{\mathbb{R}}(\partial\Omega)$, then, since Ω is a Dirichlet region, we can find a (unique) continuous extension $\hat{h}$ of h to $\overline{\Omega}$ such that $\hat{h}$ is harmonic on Ω. By the Maximum Principle,

$$h \mapsto \int_{\overline{\Omega}} \hat{h} d\mu$$

defines a positive, bounded linear functional on $C_{\mathbb{R}}(\partial\Omega)$. The Riesz Representation Theorem now provides a (unique) finite, positive Borel measure $\hat{\mu}$ with support in $\partial\Omega$ such that

$$\int_{\partial\Omega} h d\hat{\mu} = \int_{\overline{\Omega}} \hat{h} d\mu$$

for all h in $C_{\mathbb{R}}(\partial\Omega)$. The measure $\hat{\mu}$ is called the *sweep* (or *balayage*) of μ to $\partial\Omega$; cf., [6]. Choosing $h \equiv 1$ in this equation shows that $||\hat{\mu}|| = ||\mu||$. Decomposing μ as $\mu = \mu_0 + \mu_1$, where $\mu_0 := \mu|_\Omega$ and $\mu_1 := \mu|_{\partial\Omega}$, we find that $\hat{\mu} = \hat{\mu}_0 + \mu_1$, where $\hat{\mu}_0 \ll \omega(\cdot, \Omega, \alpha)$ – harmonic measure on $\partial\Omega$ for evaluation at some prescribed α in Ω. In fact, $\hat{\mu}_0$ is given by

$$d\hat{\mu}_0(\zeta) = g(\zeta) d\omega(\zeta, \Omega, \alpha),$$

where $g \in L^1(\omega(\cdot, \Omega, \alpha))$ and is defined by

$$g(\zeta) := \int_\Omega \tfrac{d\omega(\zeta,\Omega,z)}{d\omega(\zeta,\Omega,\alpha)} d\mu_0(z).$$

In the case $\Omega = \mathbb{D}$, μ is a finite, positive Borel measure with support in $\overline{\mathbb{D}}$ and $\alpha = 0$, $\hat{\mu}_0$ has the form

$$d\hat{\mu}_0 = g dm,$$

where $g \in L^1(m)$ and is defined by

$$g(\xi) := \int_{\mathbb{D}} P_w(\xi) d\mu_0(w).$$

Since, in this case, $g \in L^1(m)$, our next result is an immediate consequence of Theorem 2.2.

Corollary 2.4. *Let μ be a finite, positive Borel measure with support in $\overline{\mathbb{D}}$. Then, for m-a.a. ξ in $\mathbb{T}$, there is a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ that converges to ξ nontangentially such that*

$$\frac{\psi_\mu(z_n)}{\log|z_n|} \longrightarrow \int_{\mathbb{D}} P_w(\xi)d\mu(w),$$

as $n \to \infty$.

Proposition 2.5. *Let Ω be a Dirichlet subregion of $\mathbb{D}$, suppose $0 \in \Omega$ and let ω_Ω denote harmonic measure on $\partial\Omega$ for evaluation at 0. Then, for m-a.a. ξ in $\mathbb{T}$,*

$$\int_{\mathbb{D}} P_w(\xi)d\omega_\Omega(w) + \frac{d\omega_\Omega}{dm}(\xi) = 1.$$

Proof. Since ω_Ω is harmonic measure (on $\partial\Omega$) for evaluation at 0, the sweep of ω_Ω to $\mathbb{T}$ is precisely m. This tells us that $\int_{\mathbb{D}} P_w(\xi)d\omega_\Omega(w) + \frac{d\omega_\Omega}{dm}(\xi)$, which we know exists for m-a.a. ξ in $\mathbb{T}$, is in fact 1 (a.e. m). □

3. On Harmonic Measure

Let μ be a finite, positive Borel measure with compact support in $\mathbb{C}$. For z in $\mathbb{C}$, we let $p_\mu(z)$ denote the negative of the logarithmic potential of μ at z, namely,

$$\int \log|w-z|d\mu(w).$$

Let Ω be a Dirichlet subregion of $\mathbb{D}$ that contains 0 and let ω_Ω denote harmonic measure on $\partial\Omega$ for evaluation at 0. Then, for z in $\mathbb{D}$,

$$\begin{aligned} \psi_{\omega_\Omega}(z) &:= \int_{\partial\Omega} \log\left|\tfrac{w-z}{1-\bar{z}w}\right| d\omega_\Omega(w) \\ &= p_{\omega_\Omega}(z). \end{aligned}$$

Furthermore, if $z \in \mathbb{C}\backslash\Omega$, then $w \mapsto \log\left|\frac{w-z}{1-\bar{z}w}\right|$ is harmonic in Ω and so, by a standard argument involving balayage (cf., [11]), $p_{\omega_\Omega}(z) = \log|z|$.

Indeed, p_{ω_Ω} is harmonic in Ω and has boundary values $\log|z|$. Therefore,

$$g(z, \Omega, 0) := p_{\omega_\Omega}(z) - \log|z|$$

is the *Green's function* on Ω for evaluation at 0. The next result is a consequence of the subharmonicity of $w \mapsto \log|w - z|$ and the Maximum Principle.

Lemma 3.1. *Let W be Dirichlet region such that $0 \in W \subseteq \mathbb{D}$. Then $\log|z| < p_{\omega_W}(z)$ for all z in W. Moreover, if V is any Dirichlet region that contains 0 and that is properly contained in W, then $p_{\omega_V}(z) < p_{\omega_W}(z)$ for all z in W.*

Once again, let Ω be a Dirichlet subregion of $\mathbb{D}$ that contains 0 and let ω_Ω denote harmonic measure on $\partial\Omega$ for evaluation at 0. For $0 < t < 1$, let W_t denote $\{z \in \Omega : p_{\omega_\Omega}(z) > t \cdot \log|z|\}$. Clearly $t \cdot \log|z| > \log|z|$ whenever $0 < |z| < 1$ and, by Lemma 3.1, $p_{\omega_\Omega}(z) > \log|z|$ for all z in Ω. Therefore, since p_{ω_Ω} is continuous (indeed, harmonic) in Ω, we note that W_t is an open subset of Ω that contains 0. If W_t were not connected, then it would have a component that does not contain 0 and, on that component, $h(z) := p_{\omega_\Omega}(z) - t \cdot \log|z|$ would be positive, bounded and harmonic, with zero boundary values – an impossibility. Therefore, W_t is connected. If $z \in \mathbb{D} \cap \partial W_t$, then

$$p_{\omega_\Omega}(z) = t \cdot \log|z| > \log|z|$$

and hence $z \in \Omega$. From this and the fact that h is harmonic in $\Omega \setminus \{0\}$, we can apply the Minimum Principle and conclude that any component of ∂W_t that is contained in $\mathbb{D}$ is necessarily much more than a singleton set. It follows that W_t is a Dirichlet subregion of $\mathbb{D}$. We let ω_{W_t} denote harmonic measure on ∂W_t for evaluation at 0.

Lemma 3.2. *Maintaining the terminology of the above paragraph, if E is a Borel subset of $\mathbb{T}$ and $\omega_\Omega(\mathbb{E}) > 0$, then there exists c, $0 < c < 1$, such that $\omega_{W_c}(E) > 0$.*

Proof. Since $\omega_\Omega(E) > 0$, we can find a positive constant δ and a compact subset F of E such that $m(F) > 0$ and $\frac{d\omega_\Omega}{dm}(\xi) \geq \delta$ for all ξ in F. Choose c so that $1 - \delta < c < 1$, and let $W_c := \{z \in \Omega : p_{\omega_\Omega}(z) > c \cdot \log|z|\}$ – which we know is a Dirichlet subregion of $\mathbb{D}$ that contains 0. We let ω_{W_c} denote harmonic measure on ∂W_c for evaluation at 0. Then, for z in $\mathbb{D}$,

$$\begin{aligned} \psi_{\omega_{W_c}}(z) &:= \int \log\left|\tfrac{w-z}{1-\bar{z}w}\right| d\omega_{W_c}(w) \\ &= \int \log|w-z| d\omega_{W_c}(w) =: p_{\omega_{W_c}}(z). \end{aligned}$$

If $z \in \mathbb{D} \cap \partial W_c$ $(= \Omega \cap \partial W_c)$, then

$$p_{\omega_{W_c}}(z) = \log|z| = \frac{1}{c} \cdot p_{\omega_\Omega}(z).$$

And these equalities still hold when $z \in \mathbb{T} \cap \partial W_c$, since, in this case, all of these functions' values are zero. Therefore, $p_{\omega_{W_c}}$ and $\frac{1}{c} \cdot p_{\omega_\Omega}$ have the same boundary values in W_c. Since both of these functions are bounded and harmonic in W_c, we conclude that $p_{\omega_{W_c}} \equiv \frac{1}{c} \cdot p_{\omega_\Omega}$ in W_c. Now, by Theorem 2.2 and Proposition 2.5, for m-a.a. ξ in F, there is a sequence $\{z_n\}_{n=1}^\infty$ in $\mathbb{D}$ that converges to ξ nontangentially such that

$$\limsup_{n\to\infty} \frac{p_{\omega_\Omega}(z_n)}{\log|z_n|} = \limsup_{n\to\infty} \frac{\psi_{\omega_\Omega}(z_n)}{\log|z_n|} \leq 1 - \delta.$$

Thus, for any such ξ and for n sufficiently large, $z_n \in W_c$. Since $p_{\omega_{W_c}} \equiv \frac{1}{c} \cdot p_{\omega_\Omega}$ in W_c, it follows that

$$\limsup_{n\to\infty} \frac{\psi_{\omega_{W_c}}(z_n)}{\log|z_n|} = \limsup_{n\to\infty} \frac{p_{\omega_{W_c}}(z_n)}{\log|z_n|} \leq \frac{1-\delta}{c}.$$

So, by the first part of Theorem 2.2,

$$\int_{\mathbb{D}} P_w(\xi) d\omega_{W_c}(w) \leq \frac{1-\delta}{c} < 1,$$

for m-a.a. ξ in F. Applying Proposition 2.5 once more, we obtain:

$$\frac{d\omega_{W_c}}{dm}(\xi) \geq 1 - \frac{1-\delta}{c} > 0$$

for m-a.a. ξ in F. And, since $m(F) > 0$, we can now conclude that $0 < \omega_{W_c}(F) \leq \omega_{W_c}(E)$. $\square$

In preparation for our next results we recall some standard notation. If K is a compact subset of $\mathbb{C}$, then $\{z \in \mathbb{C} : |p(z)| \leq ||p||_K$ for all polynomials $p\}$ is the so-called *polynomially convex hull* of K and is denoted $\hat{K}$. So $\hat{K}$ is K with its bounded holes filled-in. In the case that Γ is a Jordan curve, $\hat{\Gamma}$ is the closure of the Jordan region whose boundary is Γ.

Theorem 3.3. *Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of smooth Jordan curves in $\mathbb{D}$ such that $\hat{\Gamma_k} \cap \hat{\Gamma_l} = \emptyset$, whenever $k \neq l$, and such that any compact subset of $\mathbb{D}$ has nonempty intersection with only finitely many of the $\Gamma_{n's}$. Then $\Omega := \mathbb{D} \setminus (\cup_{n=1}^{\infty} \hat{\Gamma_n})$ is a Dirichlet subregion of $\mathbb{D}$ which we assume, without loss of generality, contains 0. Let ω_Ω denote harmonic measure on $\partial\Omega$ for evaluation at 0. If E is a Borel subset of $\mathbb{T}$ and $\omega_\Omega(E) > 0$, then there is a simply connected region G, where $0 \in G \subseteq \mathbb{D}$, $\partial G \subseteq \Omega \cup \mathbb{T}$ and $\omega_G(E) > 0$ such that $U := \Omega \cup G$ has the property that ω_Ω and ω_U are boundedly equivalent on ∂U.*

Proof. By Lemma 3.2, there exists c, $0 < c < 1$, such that $\omega_{W_c}(E) > 0$; recall that $W_c := \{z \in \Omega : p_{\omega_\Omega}(z) > c \cdot \log |z|\}$. Let G be the smallest simply connected region that contains W_c. So, G is obtained from W_c by filling in its "holes" – the bounded components of $\mathbb{C} \setminus W_c$. Notice that $\partial G \subseteq \partial W_c \subseteq \Omega \cup \mathbb{T}$. Since $\mathbb{T} \cap \partial G = \mathbb{T} \cap \partial W_c$, $W_c \subseteq G$ and $\omega_{W_c}(\mathbb{T}) > 0$, it follows that $\omega_G(\mathbb{T}) > 0$. We let U denote the Dirichlet region $\Omega \cup G$. If $\xi \in \mathbb{T} \cap \partial G$, then we can find a sequence $\{z_n\}_{n=1}^{\infty}$ in W_c such that $\xi = \lim_{n\to\infty} z_n$; again, since $\mathbb{T} \cap \partial G = \mathbb{T} \cap \partial W_c$. And since $\{z_n\}_{n=1}^{\infty} \subseteq W_c$, we have:

$$c \cdot \log|z_n| < p_{\omega_\Omega}(z_n) = \psi_{\omega_\Omega}(z_n)$$

and hence,

$$\frac{\psi_{\omega_\Omega}(z_n)}{\log|z_n|} < c$$

for all n. The first part of Theorem 2.2 now tells us that

$$\int P_w(\xi) d\omega_\Omega(w) \leq c < 1,$$

and this holds for all ξ in $\mathbb{T} \cap \partial G$. By Proposition 2.5, we conclude that $\frac{d\omega_\Omega}{dm}(\xi) > 1 - c$ for m - a.a ξ in $\mathbb{T} \cap \partial G$. And, by the Maximum Principle, $\omega_U|_{\mathbb{T}} \leq m$. It now follows that ω_U and ω_Ω are boundedly equivalent on $\mathbb{T} \cap \partial G$. What remains to be shown is that these harmonic measures are boundedly equivalent elsewhere on ∂U; and "elsewhere" consists of $\partial U \setminus \overline{G}$. To this end we let Ω_n (for $n = 1, 2, 3, ...$) denote the finitely connected region $\mathbb{D} \setminus (\cup_{k=1}^{n} \hat{\Gamma_k})$ and let U_n denote $\Omega_n \cup G$. Clearly, $\Omega \subseteq \Omega_n$ for all n and $\Omega = \cap_{n=1}^{\infty} \Omega_n$. Hence, by Lemma 3.1, $p_{\omega_{\Omega_n}}(z) \geq p_{\omega_\Omega}(z)$ for all z in Ω, and all n. Since $p_{\omega_\Omega}(z) \geq c \cdot \log|z|$ for all z in $\Omega \cap \partial G$ $(= \mathbb{D} \cap \partial G)$ we find that, for such z,

$$g(z, \Omega_n, 0) := p_{\omega_{\Omega_n}}(z) - \log|z| \geq (c-1)\log|z|.$$

Since $\log|z| \le p_{\omega_{U_n}}(z) \le 0$ for all z in $\mathbb{C}$, we now have:

$$\begin{aligned} g(z, U_n, 0) &:= p_{\omega_{U_n}}(z) - \log|z| \\ &\le -\log|z| \\ &\le \frac{1}{1-c} \cdot g(z, \Omega_n, 0), \end{aligned}$$

for all z in $\mathbb{D} \cap \partial G$ (and $n = 1, 2, 3, ...$). Moreover, $g(\zeta, \Omega_n, 0) = g(\zeta, U_n, 0) = 0$ for all ζ in ∂U_n. Putting these things together, we find that the boundary values of $\frac{1}{1-c} \cdot g(z, \Omega_n, 0)$ dominate the boundary values of $g(z, U_n, 0)$ on each component of $\Omega_n \setminus \overline{G}$, where each of these functions are harmonic. Therefore,

$$g(z, U_n, 0) \le \frac{1}{1-c} \cdot g(z, \Omega_n, 0),$$

for all z in $\Omega_n \setminus \overline{G}$. Since Ω_n and U_n are finitely connected and have smooth boundary components, the harmonic measures on the boundaries of each for evaluation at 0 are weighted arc-length measures, where the weights are the inward normal derivatives of their respective Green's functions (for evaluation at 0). Therefore,

$$\omega_{U_n} \le \frac{1}{1-c} \cdot \omega_{\Omega_n}$$

on $\partial U_n \setminus \overline{G}$, for $n = 1, 2, 3, ...$. From this it follows that

$$\omega_U \le \frac{1}{1-c} \cdot \omega_\Omega$$

on $\partial U \setminus \overline{G}$. Clearly $\Omega \subseteq U$ and so, by the Maximum Principle, $\omega_\Omega \le \omega_U$ on ∂U. Our proof is now complete. □

Let Ω be a Dirichlet region. For any (fixed) z in Ω, let $\omega(\cdot, \Omega, z)$ denote harmonic measure on $\partial\Omega$ for evaluation at z. If $0 \in \Omega$, then we let $\omega_\Omega(\cdot)$ denote $\omega(\cdot, \Omega, 0)$. Notice that, by Harnack's Inequality, for any pair of points z and w in Ω, $\omega(\cdot, \Omega, z)$ and $\omega(\cdot, \Omega, w)$ are boundedly equivalent measures on $\partial\Omega$.

Lemma 3.4. *Let Ω be a Dirichlet subregion of $\mathbb{D}$, let E be a Borel subset of $\mathbb{T}$ and define h_E on $\mathbb{D}$ by*

$$h_E(z) = \begin{cases} \omega(E, \Omega, z) & \text{if } z \in \Omega \\ 0 & \text{if } z \in \mathbb{D} \setminus \Omega \ . \end{cases}$$

Then h_E is continuous on $\mathbb{D}$.

Proof. Let h_n be a sequence of functions that are continuous on $\overline{\Omega}$ and harmonic in Ω such that $0 \le h_n \le 1$ (on $\overline{\Omega}$), $h_n \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D} \cap \partial\Omega$ and $h_n \longrightarrow \chi_E$ a.e. m on $\mathbb{T}$ (as $n \to \infty$). Then h_n converges uniformly on compact subsets of $\mathbb{D} \cap \overline{\Omega}$ (as $n \to \infty$) and the limit there is necessarily h_E. This tells us that h_E is continuous on $\mathbb{D} \cap \overline{\Omega}$. Since h_E is identically zero on $\mathbb{D} \cap \partial\Omega$, we find that h_E is in fact continuous on $\mathbb{D}$. □

Proposition 3.5. *Let Ω be a Dirichlet region such that $0 \in \Omega \subseteq \mathbb{D}$, let F be a compact subset of $\mathbb{T}$ and suppose that $\omega_\Omega(F) > 0$. For $0 < t < \omega_\Omega(F)$, let V_t be the component of $\{z \in \Omega : \omega(F, \Omega, z) > t\}$ that contains 0. Then V_t is itself a Dirichlet region, $\mathbb{T} \cap \partial V_t \subseteq F$ and $\omega_{V_t}(F) > 0$.*

Proof. By the continuity of $z \mapsto \omega(F, \Omega, z)$ in Ω, V_t is a (nonempty) subregion of Ω. Let V_t^* be the smallest simply connected region that contains V_t – obtained from V_t by filling in its "holes". And we define a hole in V_t to be a bounded component of $\mathbb{C} \setminus V_t$. To establish that V_t is a Dirichlet region, it is sufficient to show that each hole in V_t is more than a singleton set. Let K be such a hole and let h_F be as defined in Lemma 3.4. Since h_F is positive and harmonic in Ω, by the Minimum Principle, K cannot be contained in Ω. So, K must contain some point z_0 in $\mathbb{D} \setminus \Omega$; wherefore, $h_F(z_0) = 0$. Yet, by the continuity of h_F in $\mathbb{D}$ (cf., Lemma 3.4), K necessarily contains at least one point z' such that $h_F(z') = t$ (> 0). We conclude that K is not a singleton set and indeed that V_t is a Dirichlet region. Let ω_{V_t} denote harmonic measure on ∂V_t for evaluation at 0. Notice that $z \mapsto \omega(\mathbb{T}, \Omega, z) - t$ is bounded, positive and harmonic in V_t and, by Lemma 3.4, is identically zero on $\mathbb{D} \cap \partial V_t$. This tells us that $\omega_{V_t}(\mathbb{T}) > 0$. To establish that $\mathbb{T} \cap \partial V_t \subseteq F$ we argue indirectly and suppose, to the contrary, that there exists ζ_0 in $\mathbb{T} \setminus F$ such that $\zeta_0 \in \partial V_t$. Then we can find a sequence $\{z_n\}_{n=1}^{\infty}$ in V_t such that $z_n \longrightarrow \zeta_0$, as $n \to \infty$. So, by the Maximum Principle,

$$\begin{aligned} \int_F P_{z_n}(\zeta) dm(\zeta) &= \omega(F, \mathbb{D}, z_n) \\ &\geq \omega(F, V_t, z_n) > t > 0 \end{aligned}$$

for all n. But since F is a compact subset of $\mathbb{T}$, $\zeta_0 \notin F$ and $z_n \longrightarrow \zeta_0$ (as $n \to \infty$), it follows that $P_{z_n}(\zeta) \longrightarrow 0$ uniformly for ζ in F, as $n \to \infty$. And so, necessarily, $\int_F P_{z_n}(\zeta) dm(\zeta) \longrightarrow 0$, as $n \to \infty$; which gives us a contradiction. Hence, $\mathbb{T} \cap \partial V_t \subseteq F$. $\square$

Our next theorem is the main result of the paper.

Theorem 3.6. *Let Ω be a Dirichlet region such that $0 \in \Omega \subseteq \mathbb{D}$ and let E be a Borel subset of $\mathbb{T}$. If $\omega_\Omega(E) > 0$, then there is a simply connected region G, where $0 \in G \subseteq \mathbb{D}$, $\partial G \subseteq \Omega \cup E$ and $\omega_G(E) > 0$ such that $U := \Omega \cup G$ has the property that ω_Ω and ω_U are boundedly equivalent on ∂U.*

Proof. Let Ω^* be the smallest simply connected region that contains Ω – obtained from Ω by filling in its holes – and let φ be a conformal mapping from Ω^* onto $\mathbb{D}$ such that $\varphi(0) = 0$. If we establish this theorem for $\varphi(\Omega)$ in place of Ω, then we can carry over the result (under φ^{-1} and its boundary values) to the setting of Ω. So, we may assume from the outset that $\Omega^* = \mathbb{D}$. Now, since $\omega_\Omega(E) > 0$, we can find a positive constant δ and a compact subset F of E such that $\omega_\Omega(F) > 0$ and $\frac{d\omega_\Omega}{dm}(\zeta) \geq \delta > 0$ for all ζ in F . For $0 < t < \omega_\Omega(F)$, let V_t and V_t^* be defined as in Proposition 3.5 and its proof. By Proposition 3.5, V_t is itself a Dirichlet region, $\mathbb{T} \cap \partial V_t^* = \mathbb{T} \cap \partial V_t \subseteq F$ and $\omega_{V_t}(F) > 0$. Let K be a hole in V_t; that is, a bounded component of $\mathbb{C} \setminus V_t$. As we observed in the proof of Proposition 3.5, K necessarily contains a nontrivial part of $\mathbb{D} \cap \partial\Omega$. Let h_F be as defined in Lemma 3.4. Since h_F is continuous on $\mathbb{D}$ (cf., Lemma 3.4), $h_F(z) = t$ for all z in ∂K. Therefore, since $h_F \equiv 0$ on $\mathbb{D} \cap \partial\Omega$, it follows that $K \cap \partial\Omega \subseteq K^0$ – the interior of K. Let O be a nonempty component of K^0. Then $\partial O \subseteq \partial K$ and hence $h_F(z) = t$ for all z in ∂O. If $O \subseteq \Omega$, then we can apply the Minimum and Maximum Principles and find that $\omega(F, \Omega, z) = t$ for all z in O. And from this we can infer that $\omega(F, \Omega, z) \equiv t$ in Ω; which is not possible. So O must contain a nontrivial part of $\partial\Omega$ and $O \cap \partial\Omega$ is necessarily compact; let $C = O \cap \partial\Omega$. Notice that $h_F \equiv 0$ on $\hat{C}$ – the polynomially convex hull of C. Therefore, $\hat{C}$ is a compact subset of O and hence the distance between $\hat{C}$ and ∂O is positive. By standard methods (c.f., [5], the proof of Proposition 1.1 in Chapter VIII) we can find finitely many smooth Jordan curves $\Gamma_1, \Gamma_2, ..., \Gamma_n$ in $O \cap \Omega$ such that

a) $\hat{C} \subseteq \bigcup_{k=1}^{n} \text{inside}(\Gamma_k)$,
b) $\text{distance}(z, C) < \text{distance}(z, \partial O)$ for any z in $\bigcup_{k=1}^{n} \Gamma_k$, and
c) $\text{distance}(\hat{\Gamma}_k, \hat{\Gamma}_l) \geq \frac{1}{3} \cdot \text{distance}(\hat{C}, \partial O)$, whenever $k \neq l$.

Repeat this for each of the components of K^0. Notice that K^0 can have only finitely many components since h_F is uniformly continuous on compact subsets of $\mathbb{D}$ (cf., Lemma 3.4). And since each hole in V_t has nonempty interior, V_t can have at most countably many holes. Continue this process for each hole in V_t and thus obtain a collection of smooth Jordan curves $\{\Gamma_\eta\}_\eta$ – at most countably many in all – such that

1) $V_t^* \cap \partial\Omega \subseteq \bigcup_\eta \text{inside}(\Gamma_\eta)$,
2) $\hat{\Gamma}_{\eta_1} \cap \hat{\Gamma}_{\eta_2} = \emptyset$, whenever $\eta_1 \neq \eta_2$, and
3) Every $\hat{\Gamma}_\eta$ is contained in some hole in V_t.

Furthermore, on any compact subset of $\mathbb{D}$, there is a positive lower bound on the distance between any pair of these curves; again, as a consequence of Lemma 3.4 and our construction per hole. Therefore, $\Lambda := V_t^* \setminus (\bigcup_\eta \hat{\Gamma}_\eta)$ is a Dirichlet region, and $V_t \subseteq \Lambda \subseteq V_t^*$. So, by the Maximum Principle, $\omega_\Lambda(F) > 0$. Let ψ be a conformal mapping from V_t^* onto $\mathbb{D}$ such that $\psi(0) = 0$ and let $D = \psi(\Lambda)$. Notice that D is a region of the type given in the statement of Theorem 3.3. Moreover, under the nontangential boundary values of ψ^{-1}, which we denote by

$\tilde{\psi}^{-1}$, we can find a compact subset J of $\mathbb{T}$ such that $\omega_D(J) > 0$ and $\tilde{\psi}^{-1}(J) \subseteq F$. We now apply the proof of Theorem 3.3 in this setting of D and J to find a constant c $(0 < c < 1)$ and a simply connected region S such that $0 \in S \subseteq \mathbb{D}$, $\partial S \subseteq D \cup \mathbb{T}$, $\omega_S(J) > 0$ and $p_D(w) \geq c \cdot \log|w|$ for all w in $D \cap \partial S$. Carrying this over under ψ^{-1}, we have a simply connected region G such that $0 \in G \subseteq \mathbb{D}$, $\partial G \subseteq \Omega \cup E$, $\omega_G(E) > 0$ and $p_\Lambda(z) \geq c \cdot \log|z|$ for all z in $\Omega \cap \partial G$. We now build a generic region, of the type described in Theorem 3.3, that contains Λ and is contained in Ω. First we note that $\mathbb{D} \cap \partial\Omega \subseteq \mathbb{D} \setminus \overline{\Lambda}$, and that any component of $\mathbb{D} \cap \partial\Omega$ has positive distance to $\overline{\Lambda}$. And recall that $\Omega^* = \mathbb{D}$. So, using the absolute continuity of h_F on compact subsets of $\mathbb{D}$, we can find at most countably many smooth Jordan curves $\sigma := \{\gamma_n\}_n$ in $\mathbb{D} \setminus \overline{\Lambda}$ such that

i) $\hat{\gamma_n} \cap \hat{\gamma_m} = \emptyset$ if $m \neq n$,
ii) any compact subset of D intersects only finitely many $\gamma_{n's}$ and
iii) $\Omega_\sigma := \mathbb{D} \setminus (\cup_n \hat{\gamma_n})$ contains Λ and is contained in Ω.

Therefore, Ω_σ is itself a Dirichlet region and, by Lemma 3.1, $p_{\Omega_\sigma}(z) \geq c \cdot \log|z|$ for all z in ∂G. Hence, if $U_\sigma := G \cup \Omega_\sigma$ and $M := \max(\frac{1}{\delta}, \frac{1}{1-c})$, then, by the proof of Theorem 3.3 (and the Maximum Principle),

$$\omega_{\Omega_\sigma} \leq \omega_{U_\sigma} \leq M \cdot \omega_{\Omega_\sigma}$$

on ∂U_σ. Since c is independent of σ, our goal follows (cf., notes on balayage in [11]). Indeed, if $U := G \cup \Omega$, then

$$\omega_\Omega \leq \omega_U \leq M \cdot \omega_\Omega$$

on ∂U. $\square$

Remark 3.7. Let μ be a finite, positive Borel measure with support in $\overline{\mathbb{D}}$ such that $P^t(\mu)$ – the closure of the polynomials in $L^t(\mu)$ $(1 \leq t < \infty)$ – is irreducible; that is, $P^t(\mu)$ contains no nontrivial characteristic functions. Consequences of this assumption are that every function f in $P^t(\mu)$ has an analytic continuation $\hat{f}$ to $\mathbb{D}$ and $\mu|_{\mathbb{T}} \ll m$. If $\mu(\mathbb{T}) > 0$, then A. Aleman, S. Richter and C. Sundberg have recently shown that $\dim(\mathcal{M}/z\mathcal{M}) = 1$ for each nontrivial, closed invariant subspace $\mathcal{M}$ for the shift on $P^t(\mu)$. In the process they also show that if $f \in P^t(\mu)$, then $\hat{f}$ has nontangential boundary values a.e. $\mu|_{\mathbb{T}}$ that agree with f; cf., [3]. If μ is a Jensen measure on $\overline{\mathbb{D}}$ and every function f in $P^t(\mu)$ has an analytic continuation to $\mathbb{D}$, then it follows that $P^t(\mu)$ is irreducible. And therefore, in this case, if $\mu(\mathbb{T}) > 0$, then [3] applies. Since harmonic measure on a Dirichlet subregion Ω of $\mathbb{D}$ is a Jensen measure, [3] speaks to the context of this paper. We observe that Theorem 3.6 (in this paper) gives an alternate proof of the result of [3] in the special case that ω_Ω is harmonic measure on the boundary of some Dirichlet region Ω, where $\Omega \subseteq \mathbb{D}$, $\omega_\Omega(\mathbb{T}) > 0$ and every function f in $P^t(\omega_\Omega)$ has an analytic continuation to $\mathbb{D}$. Indeed, by Theorem 3.6, if E is any Borel subset of $\mathbb{T}$ and $\omega_\Omega(E) > 0$, then there is a simply connected region G, where $0 \in G \subseteq \mathbb{D}$, $\partial G \subseteq \Omega \cup E$ and $\omega_G(E) > 0$ such that the mapping,

$$f \mapsto \hat{f}|_G$$

is a bounded operator from $P^t(\omega_\Omega)$ into the Hardy space $H^t(G)$; in the terminology of [2], ω_Ω is *strongly inscribed*. Nevertheless, the primary import of Theorem 3.6 is potential theoretic in nature and gives much more information than is needed for this application concerning the index of the shift. We mention another application in the context of so-called *champagne subregions*; cf., [7], [9] and [1]. A consequence of [7], Theorem 1 is that, if Ω is a champagne subregion of $\mathbb{D}$ (with certain restrictions on the spacing and size of the bubbles) and if $\omega_\Omega(\mathbb{T}) > 0$, then the distribution of ω_Ω on any bubble is basically is independent of its distribution on the other bubbles and hence one can remove large quantities of bubbles without effectively changing the distribution of ω_Ω on the remaining bubbles, or on $\mathbb{T}$. The restrictions (in [7]) on the bubbles of Ω force the result to be global in nature. Theorem 3.6 shows that, even in the most general setting, a local result still holds.

References

[1] J. R. Akeroyd, Champagne subregions of the disk whose bubbles carry harmonic measure, Math. Ann. (323), no. 2, (2002), 267-279.

[2] J. R. Akeroyd, Another look at some index theorems for the shift, Indiana Univ. Math. J. (50), no. 2, (2001), 705-718.

[3] A. Aleman, S. Richter, C. Sundberg, Nontangential limits in $P^t(\mu)$-spaces and the index of invariant subspaces, preprint.

[4] C. J. Bishop, L. Carleson, J. B. Garnett, P. W. Jones, Harmonic measures supported on curves, Pacific J. Math. 138 (1989), no. 2, 233-236.

[5] J. B. Conway, *Functions of One Complex Variable*, Springer-Verlag, 1973.

[6] J. B. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monographs, Vol. 36 (1991), Amer. Math. Soc., Providence, RI.

[7] J. Ortega-Cerda, K. Seip, Harmonic measure and uniform densities, Indiana Univ. Math. J. (53), no. 3, (2004), 905-924.

[8] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[9] J. B. Garnett, F. W. Gehring, P. W. Jones, Conformally invariant length sums, Indiana Univ. Math. J. (32), no. 6, (1983), 809-829.

[10] Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer-Verlag, 1992.

[11] M. Tsuji, *Potential Theory in Modern Function Theory* (second edition), Chelsea, New York, 1975.

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