30-th *Annual Spring Lecture Series in the Mathematical Sciences*

Commutative Algebra: Recent Developments in Tight Closure

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Science Engineering Building 403, University of Arkansas

ABSTRACTS.

Tight closure and plus closure in dimension two

HOLGER BRENNER, University of Sheffield, UK

We prove that for a homogeneous ideal in a two-dimensional standard-graded domain over a finite field of positive characteristic tight closure and plus closure are the same. This result relies on a third equivalent numerical condition in terms of the strong Harder-Narasimhan filtration of the syzygy bundle for ideal generators on the corresponding projective curve.

The Radu-André homomorphisms and their applications

FLORIAN ENESCU, Georgia State University, USA

Given two rings R and S of characteristic $p > 0$ and a homomorphism $\varphi : R \to S$, one can naturally define the so-called Radu-André ring homomorphisms. They can be used in studying positive characteristic phenomena that are shared by R and S via φ. The idea originates in work of Grothendieck and was extensively studied by Radu and André, and later Dumitrescu. We will present the theory of these homomorphisms and discuss ways they have appeared in tight closure theory, with reference to work of Hochster-Huneke, Lyubeznik-Smith, Hashimoto and myself.

Special parts of closures and analogues of analytic spread

NEIL EPSTEIN, University of Kansas, USA

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Given a closure operation c, one can sometimes define a *special part* $c$sp of c, which is required to satisfy a set of four natural axioms. If so, then every proper ideal has a minimal c-reduction. If moreover a matroid-theoretic lemma holds for c and $c$sp on I, then all minimal c-reductions of an I have the same size minimal generating set, called the *c-spread* of I (by analogy with analytic spread $\ell(I)$ for integral closure), written $\ell^c(I)$. Since all these things hold under quite general circumstances for Frobenius, tight, and plus closures, we arrive at the new invariants $\ell^F(I)$, $\ell^*(I)$, and $\ell^+(I)$. I discuss their meanings, including some joint work with Adela Vraciu on $\ell^*(I)$. Finally I present a "special" version of the Briançon-Skoda Theorem.

Generalized test ideals and jumping exponents

Nobuo Hara, Tohoku University, Japan

Given any nonzero ideal I of a regular local ring R of characteristic $p > 0$, the generalized test ideals $\tau(I^t)$ with real exponent $t > 0$ are defined and enjoy interesting properties similar to multiplier ideals. A jumping exponent is a critical exponent c such that $\tau(I^t) \neq \tau(I^c)$ for all $t < c$. Based on Monsky's idea, we will discuss the rationality and the discreteness of jumping exponents in the simplest non-trivial case.

Generators of D-modules in characteristic p > 0

Gennady Lyubeznick, University of Minnesota, USA

Let k be a field, let R be the ring of polynomials or formal power series over k, let D be the ring of k-linear differential operators and let f be an element of R. We prove that if the characteristic of k is $p > 0$, then R_f is generated as a D-module by $1/f$. This is an amazing fact considering that the corresponding characteristic zero statement is very false. We show that this fact about D-modules has a potential for applications in the theory of tight closure.

Tight closure test exponents for certain parameter ideals

Rodney Sharp, University of Sheffield, UK

This talk is concerned with the tight closure of an ideal I in a commutative Noetherian ring R of prime characteristic p. The formal definition requires, on the face of things, an infinite number of checks to determine whether or not an element of R belongs to the tight closure of I. The situation in this respect is much improved by Hochster's and Huneke's test elements for tight closure, which exist when R is a reduced algebra of finite type over an excellent local ring of characteristic p.

More recently, Hochster and Huneke have introduced the concept of test exponent for tight closure: existence of these test exponents would mean that one would have to perform just one single check to determine whether or not an element of R belongs to the tight closure of I. However, to quote Hochster and Huneke, 'it is not at all clear whether to expect test exponents to exist; roughly speaking, test exponents exist if and only if tight closure commutes with localization'.

The purpose of this talk is to outline a short direct proof that test exponents exist for parameter ideals in a reduced excellent equidimensional local ring of characteristic p.

Pure subrings of regular local rings are pseudo-rational

Hans Schoutens, NYC College of Technology (CUNY), USA

Let R be a pure subring of a regular local ring containing a field. Hochster-Roberts were the first to show that such a ring is Cohen-Macaulay, using prime characteristic methods and reduction. Later an elegant tight closure proof was given by Hochster-Huneke (as well as a proof via big CM algebras). Boutot, shortly after the H-R proof, showed, using deep vanishing theorems, that when everything is of finite type over a field of characteristic zero, then in fact R has rational singularities. Smith was able to prove the positive characteristic analogue of this, without any finiteness condition, replacing 'rational singularities' by 'pseudo-rationality'–a characteristic-free (and resolution-free) notion introduced by Lipman-Teissier. She used a notion from tight closure to do this: F-rationality. Hara, building on her work, showed that in characteristic zero, pseudo-rationality is the same as being of F-rational type. Unfortunately, this characterization falls short of yielding an easier proof of Boutot's result. I gave an alternative characterization of rational singularities of finite type over a field, in terms of the non-standard Frobenius, that does allow for an easy deduction of Boutot's result. In this talk, I will generalize all this by dropping any finite type condition: R as above is always pseudo-rational. The proof uses non-standard tight closure, in its general form due to Aschenbrenner and myself, together with the theory of 'local ultra-cohomology', which is a generalization of local cohomology, encoding the local cohomology of the reductions as well.

F-regularity and F-rationality in multigraded rings

Anurag Singh, Georgia Tech University, USA

Let R be an $\mathbb{N}^2$-graded ring over a field, and $\Delta = (m,n)\mathbb{Z}$ denote the (m,n)-diagonal in $\mathbb{Z}^2$. The *diagonal subalgebra* R_Δ is the subring $\oplus_{i\geq 0} R_{(im,in)}$ of R. Such subalgebras arise naturally as homogeneous coordinate rings of blow-ups of projective varieties. We shall discuss the properties of F-rationality, F-regularity, and rational singularities in multigraded rings and their diagonal subalgebras. As a consequence of some of our results, we show that there exist standard bigraded hypersurfaces whose rings of invariants under torus actions have rational singularities, but are not of F-regular type. The talk is based on joint work with Kei-ichi Watanabe.

A characteristic p analogue of a subadditvity property of multiplier ideals and its application

Shunsuke Takagi, Kyushu University, Japan

Demailly, Ein and Lazarsfeld proved a subadditivity property of multiplier ideals on smooth varieties, which states that the multiplier ideal of the product of two ideals is contained in the product of their individual multiplier ideals. But several counterexamples to their formula are known on singular varieties. In this talk, I will give a generalization of their formula to the case of singular varieties by using the theory of a generalization of tight closure, which is introduced by Hara and Yoshida. As an application of my generalization, I can improve Hochster-Huneke's result on the growth of symbolic powers of ideals in singular affine algebras.

The F-rational signature of a local ring of characteristic p

Yongwei Yao, University of Michigan, USA

Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring of prime characteristic p. Recall that R is defined to be F-rational if $I^* = I$ for every ideal I generated by a system of parameters. As tight closure may be characterized by Hilbert-Kunz multiplicity, we define the *F-rational signature* of R to be the infimum of $e_{HK}(I, R) - e_{HK}(J, R)$ in which $I \subsetneq J$ and I runs over all ideals generated by systems of parameters.

It is immediate that a positive F-rational signature implies F-rationality. What is more interesting is that the converse is true, at least when R is excellent. The proof of the converse relies on the following behavior of Hilbert-Kunz multiplicity: For any given $\mathfrak{m}$-primary ideal I, there exists a real number $\delta > 0$ such that, for any J containing I, $e_{HK}(I, R) - e_{HK}(J, R)$ is either 0 or $\geq \delta$. The result presented in this talk is joint work with M. Hochster.