Principal Minor Ideals with Matroid Theory

Special Session on Combinatorial and Computational Commutative Algebra and Algebraic Geometry
AMS Spring Western Sectional, University of Utah

Ashley K. Wheeler
University of Arkansas, Fayetteville
comp.uark.edu/~ashleykw

9 April, 2016
Table of Contents

1. Ideals Generated by Principal Minors
   Size $t = 2$ Principal Minors
   Size $t = n - 1$ Principal Minors
   Size $t = n - 2$ Principal Minors: Rank $r = n - 2$ Case

2. Connection to Matroid Theory
   $K$-Representable Matroids
   Matroid Subvarieties of a Grassmannian
   Positroid Varieties
Ideals Generated by Principal Minors

Connection to Matroid Theory

Size $t = 2$ Principal Minors
Size $t = n - 1$ Principal Minors
Size $t = n - 2$ Principal Minors: Rank $r = n - 2$ Case

---

Ideals Generated by Principal Minors

Thank-you for the invitation to speak!

$r, s$ arbitrary positive integers

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1s} \\ \vdots & \ddots & \ddots & \vdots \\ x_{r1} & \cdots & \cdots & x_{rs} \end{pmatrix}$$

$K[X] = \text{polynomial ring over } K \text{ with variables } x_{11}, \ldots, x_{rs}$

The **principal** minors of an $n \times n$ matrix are those whose defining row and column indices are the same.

$\mathfrak{B}_t = \text{ideal in } K[X] \text{ generated by the size } t \text{ principal minors of the generic square matrix } X$
Theorem (–)

For all $n$, $K[X]/\mathcal{B}_2$ is a complete intersection, is isomorphic to a $K$-algebra generated by monomials, and is normal. In particular, it is strongly $F$-regular (characteristic $p > 0$ case) and Gorenstein.

The proof exploits the fact that $\mathcal{B}_2$ is toric. For $t > 2$ it becomes more convenient to study components of $\mathcal{V}(\mathcal{B}_t)$ by fixing their matrix rank.

$$Y_{n,r,t} = \mathcal{V}(\mathcal{B}_t) \bigcap \{n \times n \text{ matrices of rank } r\}$$
Size $t = n - 1$ Principal Minors

**Lemma (–)**

*In the localized ring $K[X]_{\det X}$, the $K$-algebra automorphism $X \to X^{-1}$ induces an isomorphism of the schemes defined, respectively, by $\mathcal{B}_t \cdot K[X]_{\det X}$ and $\mathcal{B}_{n-t} \cdot K[X]_{\det X}$.***

**Theorem (–)**

*For $n \geq 4$, the minimal primes for $\mathcal{B}_{n-1}$ are the determinantal ideal $\mathfrak{I}_{n-1}$ and the contraction of $\ker \phi$ to $K[X]$, which we denote by $\mathcal{Q}_{n-1}$, where $\phi$ is the ring homomorphism

$$
\phi : K[X]_{\det X} \to \left( \frac{K[X]}{\mathcal{B}_1} \right)_{\det X}
$$

$$
X \mapsto (\det X) \cdot X^{-1}
$$

Ashley K. Wheeler  Principal minor ideals with matroid theory.
A quick corollary: $\text{ht} (\mathfrak{B}_t) \leq \binom{n+1}{2} - \binom{t+2}{2} + 4$ for $n \neq 3$.

An even more immediate corollary: $\text{ht} (\mathcal{Q}_{n-1}) = n$. Consequently, principal minor ideals are generally not Cohen-Macaulay. By Hochster+Roberts, it follows that, in particular, their quotients cannot be rings of invariants.

Note, the two components of $\mathcal{V}(\mathfrak{B}_{n-1})$ are:

1. $\mathcal{V}(\mathfrak{I}_{n-1}) = \bigcup_{r' < n-1} \mathcal{Y}_{n,r',n-1}$
2. $\mathcal{V}(\mathcal{Q}_{n-1}) = \overline{\mathcal{Y}}_{n,n-1,n-1} \supset \mathcal{Y}_{n,n-1,n-1}$
Size $t = n - 2$ Principal Minors: Rank $r = n - 2$ Case

When $t \neq 1, 2, n - 1, n$ identifying the components of $\mathcal{V}(\mathcal{B}_t)$ becomes harder.

Theorem (–)

$$\dim \mathcal{Y}_{n,n-2,n-2} = n^2 - 4 - n$$

Note, a matrix of rank $r$ can be decomposed as a product of two matrices, so we can identify $\mathcal{Y}_{n,r,r}$ with a product of two Grassmann varieties.

Let $\mathcal{G} = \text{Grass}(n - 2, n)$. 

$$A \in \mathcal{Y}_{n,r,r}$$

$$(\text{col } A, \text{row } A) \in \text{Grass}_K(r, n) \times \text{Grass}_K(r, n)$$
Given $g \in \mathcal{G}$, construct $\text{Graph}(g)$ as follows: a vertex represents an index; an edge joining two vertices indicates the Plücker coordinate with complementary indices vanishes.

**Example**

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 & \circlearrowright c_{13} & 0 & \circlearrowright c_{15} \\
0 & 1 & \circlearrowright c_{23} & 0 & \circlearrowright c_{25} \\
0 & 0 & \circlearrowright c_{33} & 1 & \circlearrowright c_{35}
\end{pmatrix}
$$
Proposition

Graph (g) is well-defined.

Given a graph G, if there exists g ∈ G such that Graph (g) = G then G is called permissible. A subvariety S ⊆ G that is the set of all points with the same permissible graph is denoted Graph (S).

Theorem (–)

A product S × T of permissible subvarieties corresponds to a component of Y_{n,n−2,n−2}. Furthermore (modulo transposition of S and T),

(a) Graph (S) is the union of a complete graph of order a > 1 and n − a isolated vertices;
(b) Graph (T) is the complement of Graph (S).
Theorem (−)

Suppose $S \times T$ is a permissible pair and Graph$(S)$ has a maximal complete subgraph of order $a$. Then

(a) $\text{codim } S = a - 1$,
(b) $\text{codim } T = 2(n - a)$, and
(c) (corollary) $2 \leq a \leq n - 1$. It follows that the minimal codimension of such $S \times T$ is $n$.

Example (Permissible Pairs for $n = 5$)

Graph$(S)$       Graph$(T)$

(1) \hspace{1cm} (2) \hspace{1cm} (3)

$\text{codim}(S \times T) = 7$ \hspace{1cm} $\text{codim}(S \times T) = 6$ \hspace{1cm} $\text{codim}(S \times T) = 5$
Connection to Matroid Theory

Matroids are a type of combinatorial data used to describe many seemingly unrelated objects in mathematics, including graphs, transversals, vector spaces, and networks. Matroid has many equivalent definitions.

Definition (Independence Axioms)

Let $E$ denote a finite set and $2^E$ its power set. Suppose $I \subseteq 2^E$. Then the system $\mathcal{M} = (E, I)$ is a matroid if and only if

1. $\emptyset \in I$,
2. if $S \in I$ and $T \subseteq S$ then $T \in I$, and
3. (Independence Augmentation Axiom) if $S, T \in I$ and $|S| > |T|$, then there exists $e \in S \setminus T$ such that $T \cup \{e\} \in I$. 

Ashley K. Wheeler
Principal minor ideals with matroid theory.
A matroid defined by a $K$-vector space is called $K$-representable. Let $A$ denote an $r \times n$ matrix and put

\[ E = \{ \text{columns of } A \} \]
\[ J = \{ \text{collections of linearly independent columns} \} \]
\[ D = 2^E \setminus J \]
\[ B = \{ \text{sets in } J \text{ with maximal cardinality} \} \]

**Question**

The Independence Augmentation Axiom implies all maximal sets in $J$ have the same cardinality. What is it? To what do $D, B$ correspond?
Using correspondingly prescribed axioms, the matroid $\mathcal{M} = (E, I)$ can also be defined using $\mathcal{D}$ and $\mathcal{B}$. Another equivalent definition:

**Definition (Rank Axioms)**

A function $r : 2^E \to \mathbb{Z}_+$ is the **rank function** of a matroid $\mathcal{M} = (E, r)$ if and only if for all $S, T \subseteq E$

(a) $0 \leq r(S) \leq |S|$,  
(b) if $T \subseteq S$ then $r(T) \leq r(S)$, and
(c) (**Submodularity**) $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$. 

Ashley K. Wheeler
Principal minor ideals with matroid theory.
Fix $r < n$. We get a matroid structure on the finite set of columns of a generic $r \times n$ matrix when we prescribe a subset of Plücker coordinates to vanish; let $D$ denote the set of indices for the vanishing Plücker coordinates.
Given such a matroid $\mathcal{M}$, the **open matroid variety** is the subset of points in $\mathcal{G} = \text{Grass}(r, n)$ whose matroid is $\mathcal{M}$. Its Zariski closure is called a **matroid variety**, which we shall denote by $\mathcal{V}(\mathcal{M})$.

**Example**

Schubert and Richardson varieties are matroid varieties.

The following example shows we cannot, in general, simply use the indices from $\mathcal{D}$ on the Plücker variables to generate the defining ideal for $\mathcal{V}(\mathcal{M})$. For any Plücker coordinate with index $i$, let $x_i$ denote the correspondingly indexed variable in the homogeneous coordinate ring for $\mathcal{G}$. 
Example (Ford)

Put $r = 3, n = 7$, and $\mathcal{D} = \{\{1, 2, 7\}, \{3, 4, 7\}, \{5, 6, 7\}\}$, the set of indices for Plücker coordinates we require to vanish. One hopes the defining ideal for $V(E, I)$ is

$$I = (x_{\{1,4,7\}}, x_{\{3,4,7\}}, x_{\{5,6,7\}}).$$

However, the defining ideal is actually

$$J = I + (x_{\{1,2,4\}}x_{\{3,5,6\}} - x_{\{1,2,3\}}x_{\{4,5,6\}}).$$
A particular class of matroid varieties exists, however, where the geometry is better behaved. A **positroid** is a matroid determined by a rank condition on cyclic intervals in $E = \{1, \ldots, n\}$, where a cyclic interval is an ordinary interval or its complement.

**Positroid varieties** are the matroid varieties we get from positroids.
Theorem (Knutson+Lam+Speyer)

Positroid varieties are normal, Cohen-Macaulay, have rational singularities, and their defining ideals are given by Plücker variables.

Theorem (–)

If an irreducible algebraic set is defined by Plücker variables for Grass $(n - 2, n)$ then it is a positroid variety.
Question (Current Work)

What about for $\text{Grass}(r, n)$ for general $r$? If irreducible algebraic subsets defined by Plücker variables are positroidal, it will follow that the components of $\mathcal{Y}_{n,r,r} \subset \mathcal{V}(\mathcal{B}_r)$ are normal, Cohen-Macaulay, and have rational singularities.
Proposition

Let $R = K[\wedge^r X] \subset K[X]$ and suppose $P \subset R$ is a prime ideal. Then the $r \times r$ minors in $P$ give a rank $r$ representable matroid.

Question

What are the conditions for two prime ideals in $K[\wedge^r X]$ to minimally cover the homogeneous maximal ideal? When is it possible, if ever, to partition the entries of $\wedge^r X$ so that the respective ideals they generate are prime?