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Heat equations in $\mathbb{R}\times\mathbb{C}$

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Abstract

Let $p:\mathbb{C}\to\mathbb{R}$ be a subharmonic, nonharmonic polynomial and $\tau\in\mathbb{R}$ a parameter. Define $\bar{Z}_{\tau p}=\frac{\partial}{\partial\bar{z}}+\tau\frac{\partial p}{\partial\bar{z}}$, a closed, densely-defined operator on $L^2(\mathbb{C})$. If $\Box_{\tau p}=\bar{Z}_{\tau p}\bar{Z}^*_{\tau p}$ and $\tau>0$, we solve the heat equation $\frac{\partial u}{\partial s}+\Box_{\tau p}u=0$, $u(0,z)=f(z)$, on $(0,\infty)\times\mathbb{C}$. The solution comes via the heat semigroup $e^{-s\Box_{\tau p}}$, and we show that $u(s,z)=e^{-s\Box_{\tau p}}[f](z)=\int_{\mathbb{C}}H_{\tau p}(s,z,w)f(w)\,dw$. We prove that $H_{\tau p}$ is C^∞ off the diagonal $\{(s,z,w)\colon s=0 \text{ and } z=w\}$ and that $H_{\tau p}$ and its derivatives have exponential decay. In particular, we give new estimates for the long time behavior of the heat equation.

Keywords: Heat kernel; Weighted $\bar\partial$; Finite type; Exponential decay; Gaussian decay; OPF operators; Polynomial model; Weakly pseudoconvex domain

1. Introduction

Let $p:\mathbb{C}\to\mathbb{R}$ be a subharmonic, nonharmonic polynomial and $\tau\in\mathbb{R}$ a parameter. If $z=x_1+ix_2$ and $\frac{\partial}{\partial\bar{z}}=\frac{1}{2}(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2})$, define $\bar{Z}_{\tau p}$ to be the operator

$$\bar{Z}_{\tau p}=\frac{\partial}{\partial\bar{z}}+\tau\frac{\partial p}{\partial\bar{z}},$$

and let $Z_{\tau p}=-\bar{Z}^*_{\tau p}=\frac{\partial}{\partial z}-\tau\frac{\partial p}{\partial z}$ be the negative of the formal L^2-adjoint of $\bar{Z}_{\tau p}$. If $\Box_{\tau p}=-\bar{Z}_{\tau p}Z_{\tau p}$, then our goal is to understand the heat equation:

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$$\begin{cases} \dfrac{\partial u}{\partial s} + \Box_{\tau p} u = 0, \\ u(0, z) = f(z). \end{cases} \tag{1}$$

We show that the solution $u(s, z)$ of (1) can be realized as an integral against a distributional kernel. Specifically, we will find a solution to (1) of the form:

$$u(s, z) = \int_{\mathbb{C}} H_{\tau p}(s, z, w) f(w)\, dw,$$

and our goal is to understand the regularity and pointwise bounds of $H_{\tau p}$ and its derivatives. We show that the heat kernel $H_{\tau p}(s, z, w)$ is smooth away from the diagonal $\{(s, z, w)\colon s = 0, z = w\}$ and our main result is as follows.

Theorem 1. *Let p be a subharmonic, nonharmonic polynomial and $\tau > 0$ a parameter. If $n \geqslant 0$ and Y^α is a product of $|\alpha|$ operators $Y = \bar{Z}_{\tau p}$ or $Z_{\tau p}$ when acting in z and $\overline{(Z_{\tau p})}$ or $\overline{(\bar{Z}_{\tau p})}$ when acting in w, there exist constants $c, c_1 > 0$ independent of τ so that*

$$\left| \frac{\partial^n}{\partial s^n} Y^\alpha H_{\tau p}(s, z, w) \right| \leqslant c_1 \frac{1}{s^{n+\frac{1}{2}|\alpha|+1}} e^{-\frac{|z-w|^2}{32s}} e^{-c\frac{s}{\mu(z,1/\tau)^2}} e^{-c\frac{s}{\mu(w,1/\tau)^2}}.$$

Also, c can be taken with no dependence on n and α.

$\mu(z, \delta)$ is a size function from the Carnot–Carathéodory geometry on polynomial models defined in Section 2. As discussed below, in light of the work of Kurata [11], we give new estimates for the long time behavior of $H_{\tau p}$. The smoothness of $H_{\tau p}$ is expected from the work of Nagel and Stein [14] and Christ [4], though the estimates for the derivatives of $H_{\tau p}$ are new. Moreover, as a consequence of [4,14] and Fu and Straube [8], we expect the results on the heat equation to have applications to partial differential equations in several complex variables. In fact, we do obtain such applications which we now describe.

The operators $\bar{Z}_{\tau p}$ and $\Box_{\tau p}$ arise in both problems in one complex variable and several complex variables. As detailed below, $\bar{Z}_{\tau p}$ is a natural operator to consider when studying the weighted $\bar{\partial}$-equation in $\mathbb{C}$ and the $\bar{\partial}_b$-problem on polynomial models in $\mathbb{C}^2$. Also, it turns out that the eigenvalues of $\Box_{np}$ as $n \to \infty$ are important to understand the compactness of $\bar{\partial}$-Neumann operator on certain classes of Hartogs domains in $\mathbb{C}^2$.

1.1. $\bar{\partial}$ on weighted L^q spaces in $\mathbb{C}$

The interest in the weighted $\bar{\partial}$-problem in $\mathbb{C}$ begins with Hörmander's solution of the inhomogeneous Cauchy–Riemann equations on pseudoconvex domains in $\mathbb{C}^n$ [10]. A crucial estimate in Hörmander's work is that for $\Omega \subset \mathbb{C}$ with $\operatorname{diam}(\Omega) \leqslant 1$, there is a solution u to $\bar{\partial}u = f$ in $L^2(\Omega, e^{-2p})$ satisfying the estimate $\int_\Omega |u|^2 e^{-2p}\, dz \leqslant \int_\Omega |f|^2 e^{-2p}\, dz$. Fornæss and Sibony [7] generalize Hörmander's weighted L^2 estimate to L^q, $1 < q \leqslant 2$. They show $\bar{\partial}u = f$ has a solution satisfying $(\int_\Omega |u|^q e^{-2p}\, dz)^{1/q} \leqslant \frac{C}{p-1}(\int_\Omega |f|^q e^{-2p}\, dz)^{1/q}$. They also show that the estimate fails if $q > 2$. Berndtsson [1] builds on the work of Fornæss and Sibony by showing an L^q–L^1 result. He shows that if $1 \leqslant q < 2$, then $\bar{\partial}u = f$ has a solution so that

$(\int_\Omega (|u|^2 e^{-p})^q\,dz)^{1/q} \leqslant C_p \int_\Omega |f| e^{-p}\,dz$. Berndtsson also proves a weighted L^∞–L^q estimate when $q > 2$.

In [4], Christ recognizes that it is possible to study the $\bar\partial$-problem in $L^2(\mathbb{C}, e^{-2p})$ by working with a related operator in the unweighted space $L^2(\mathbb{C})$. If $\bar\partial\tilde u = \tilde f$ and both $\tilde u = e^p u$ and $\tilde f = e^p f$ are in $L^2(\mathbb{C}, e^{-2p})$, then $\frac{\partial \tilde u}{\partial \bar z} = \tilde f \Leftrightarrow e^{-p}\frac{\partial}{\partial \bar z} e^p u = f$. However, $e^{-p}\frac{\partial}{\partial \bar z} e^p u = \bar Z_p u$, so the $\bar\partial$-problem on $L^2(\mathbb{C}, e^{-2p})$ is equivalent to the $\bar Z_p$-problem, $\bar Z_p u = f$, on $L^2(\mathbb{C})$. Christ solves the $\bar Z_p$-equation, $\bar Z_p u = f$, in $L^2(\mathbb{C})$. Christ proves that $G_p = \Box_p^{-1}$ is a well-defined, bounded, linear operator on $L^2(\mathbb{C})$. $R_p = Z_p G_p$ is the relative fundamental solution of $\bar Z_p$, i.e. the operator R_p satisfies $\bar Z_p R f = (I - S_p) f$ where S_p is the projection of $L^2(\mathbb{C})$ onto the $\ker \bar Z_p$. He shows that G_p and R_p can be realized as fractional integral operators with kernels $G_p(z,w)$ and $R_p(z,w)$, respectively, and he finds pointwise upper bounds on the kernels $G_p(z,w)$ and $R_p(z,w)$.

Berndtsson [3] also solves $\bar Z_p u = f$ for p subharmonic, but Berndtsson solves the problem on $L^2(\Omega)$ where $\Omega \subset \mathbb{C}$ is a smoothly bounded domain. Like Christ, he expresses his L^2-minimizing solution via a fractional integral operator, though unlike Christ, his analysis is derived through functional analysis and a careful study of Kato's inequality: $\Delta|\alpha| \geqslant \Delta p|\alpha| - 4|\Box_p \alpha|$ where $\alpha \in C^2(\Omega)$. Berndtsson views $\Box_p$ as a Schrödinger operator. Specifically, if $z = x_1 + i x_2$, then

$$2\Box_p = \frac{1}{2}(-i\nabla - a)^2 + V,$$

where $a = (-\frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_1})$ and $V = \frac{1}{2}\Delta p$. Expressed in this form, $2\Box_p$ is said to be a Schrödinger operator with magnetic potential a and electric potential V. We use this representation of $\Box_p$ in the proof of Theorem 25.

1.2. Polynomial models and Hartogs domains in $\mathbb{C}^2$

Now that we have established the connection between the weighted $\bar\partial$-equation in $\mathbb{C}$ and the operators $\bar Z_p$ and Z_p, we now turn to the study of $\bar\partial_b$-problem on polynomial models in $\mathbb{C}^2$ and their connection with the operators $\bar Z_p$ and Z_p. A polynomial model M_p is the boundary of the unbounded weakly pseudoconvex domain $\Omega_p = \{(z_1, z_2) \in \mathbb{C}^2\colon\ \operatorname{Im} z_2 > p(z_1)\}$ where p is a subharmonic, nonharmonic polynomial. Observe that the boundary $M_p \cong \mathbb{C} \times \mathbb{R}$ and the $(0,1)$-form $\bar\partial_b$ can be identified with the vector field $\bar L = \frac{\partial}{\partial \bar z_1} - 2i \frac{\partial p}{\partial \bar z_1}\frac{\partial}{\partial \bar z_2}$. Under the isomorphism, $\bar\partial_b$ (defined on M_p) becomes the vector field (still called $\bar L$ by an abuse of notation)

$$\bar L = \frac{\partial}{\partial \bar z} - i\frac{\partial p}{\partial \bar z}\frac{\partial}{\partial t}$$

defined on $\mathbb{C} \times \mathbb{R}$. There are a number of approaches that one can take to study the $\bar L$-problem. One is to take a partial Fourier transform in t because $\bar L$ is translation invariant. Under the partial Fourier transform, the vector field $\bar L$ becomes $\bar Z_{\tau p} = \frac{\partial}{\partial \bar z} + \tau\frac{\partial p}{\partial \bar z}$, which we regard as a one-parameter family of differential operators on $\mathbb{C}$ indexed by τ. Thus, questions about the $\bar\partial_b$-complex on M are closely connected with the $\bar\partial$-equation on weighted L^2-spaces in $\mathbb{C}$.

To analyze operators on Hartogs domains in $\mathbb{C}^n$, mathematicians have recognized that it is often enough to understand weighted operators on the base space and reconstruct the original operator via Fourier series [2,7,12]. Recently, on a class of Hartogs domains $\Omega \subset \mathbb{C}^2$, Fu and

Straube [8,9] establish an equivalence between the compactness of the $\bar{\partial}$-Neumann operator and the blowup of the minimal eigenvalue of $\Box_{\tau p}$ as $\tau \to \infty$. Christ and Fu [5] use the work of Fu and Straube to show the equivalence of: compactness of the solving operator of the $\bar{\partial}$-Neumann Laplacian, compactness of the solving operator of Kohn Laplacian $\Box_b$, and $b\Omega$ satisfying property (P).

1.3. Heat semigroups and heat kernels

Like Christ, we are interested in inverting $\Box_{\tau p} = -\bar{Z}_{\tau p} Z_{\tau p}$. For an alternative to Christ's approach, we can look at the heat semigroup $e^{-s\,\Box_{\tau p}}$ and integrate in s. Formally, $u = e^{-s\,\Box_{\tau p}}[f]$ solves the heat equation (1) and inverts $\Box_{\tau p}$ since

$$\int_0^\infty e^{-s\,\Box_{\tau p}}\,ds = \Box_{\tau p}^{-1} \tag{2}$$

and $u(0, z) = e^{-0\,\Box_{\tau p}}[f](z) = f(z)$.

On M, Nagel and Stein [14] investigate the heat semigroup $e^{-s\Box_b}$ to solve the heat equation $\frac{\partial u}{\partial s} + \Box_b u = 0$, with initial condition $u(0, a) = f(a)$. Their goal is to use estimates of the heat semigroup on $M \cong \mathbb{C} \times \mathbb{R}$ to understand $\Box_b$ in a product setting [15]. Nagel and Stein define $e^{-s\Box_b}$ with the spectral theorem use the Riesz Representation Theorem to write $e^{-s\Box_b}[f](\alpha) = \int_{\mathbb{C}\times\mathbb{R}} H(s, \alpha, \beta) f(\beta)\,d\beta$. H is a distributional kernel with a nonintegrable singularity when $s = 0$ and $\alpha = \beta$, and H is smooth off of the diagonal. They also obtain pointwise estimates on $H(s, \alpha, \beta)$ and its derivatives. A fundamental tool in their argument is the class of nonisotropic smoothing (NIS) operators [14,18].

A motivation for this work is to solve the problem of Christ, i.e. invert $\Box_{\tau p}$ and find pointwise estimates on $G_{\tau p}(z, \zeta)$, using the heat semigroup $e^{-s\,\Box_{\tau p}}$ method and (2). In addition, understanding the heat equation (1) is an interesting question in its own right. We follow the ideas of [14] to prove the existence and regularity of $H_{\tau p}$. Our substitution for their NIS operators are the one-parameter families (OPF) of operators defined in [19]. There is an obstruction, however, to using the techniques of Nagel and Stein in this setting. Due to the partial Fourier transform, it appears that we cannot scale in the transformed variable. Losing the ability to scale in any variable dooms the scaling argument of Nagel and Stein. We find other techniques which allow us to bound the heat kernel and its derivatives with better decay than the scaling argument would have given.

1.4. Discussion of Theorem 1

The proof of Theorem 1 has two steps. First, we show that $e^{-s\,\Box_{\tau p}}$ is an integral operator with kernel $H_{\tau p}(s, z, w)$ that is smooth away from $\{(s, z, w)\colon z = w \text{ and } s = 0\}$. To do this, we use the ideas of [14] to develop properties of OPF operators defined in [19]. From there, still following [14], we use the spectral theorem and L^2-methods to prove smoothness of $H_{\tau p}(s, z, w)$.

The second step of our analysis is to prove pointwise estimates of $H_{\tau p}(s, z, w)$ and its derivatives. This is the content of Theorem 1, and the proof has two stages. In the first stage, we write $2\,\Box_{\tau p}$ as a Schrödinger operator, similarly to Berndtsson [3]. We use the Feynman–Kac–Itô formula [24] to show Gaussian decay for $H_{\tau p}(s, z, w)$. We show the time decay of $H_{\tau p}(s, z, w)$ with an L^2-energy argument.

The goal of the second stage is to prove pointwise bounds on the derivatives $\frac{\partial^n}{\partial s^n} Y^\alpha H_{\tau p}(s,z,w)$. The idea is to prove a local L^2-bound for $\frac{\partial^n}{\partial s^n} Y^\alpha H_{\tau p}(s,z,w)$ and its derivatives and pass to a local L^∞-bound using either a Sobolev embedding-type result, Theorem 13, or the subsolution estimation from Kurata [11, Lemma 27]. The arguments rely heavily on OPF operators and their ability to commute with derivatives.

Kurata studies heat kernels in $\mathbb{R}^n$ for Schrödinger operators of the form $L = (-i\nabla - a)^2 + V$ where $a \in C^1$ and $V \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, $V \geqslant 0$. His conditions on a and V are more general than what we consider, and he proves continuity of the heat kernel. If $\deg p = 2m$, Kurata shows the bound $|H_{\tau p}(s,z,w)| \leqslant \frac{C}{s} e^{-c_2 \frac{|z-w|^2}{s}} e^{-c_3 (\frac{s}{\mu(z,1/\tau)^2})^{1/2m}}$, a weaker result than ours. The proof of Theorem 1 exploits the specific structure of $\Box_{\tau p}$ and does not seem to generalize to Kurata's more general operators.

By integrating in s, the pointwise estimates on $H_{\tau p}(s,z,w)$ allow us to recover estimates on the fundamental solution of $\Box_{\tau p}$ and compare our work to Christ [4]. If $G_{\tau p}(z,w)$ is the fundamental solution to $\Box_{\tau p}$, we prove the following result.

Corollary 2. *Let $G_{\tau p}(z,w)$ be the integral kernel of the fundamental solution for $\Box_{\tau p}$. If X^α is a product of $|\alpha|$ operators of the form $X^j = \bar{Z}_{\tau p}, Z_{\tau p}$ if acting in z and $\overline{(\bar{Z}_{\tau p})}, \overline{(Z_{\tau p})}$ if acting in w, then there exists constants $C_{1,|\alpha|}, C_2 > 0$ so that if $\tau > 0$,*

$$\left|X^\alpha G_{\tau p}(z,w)\right| \leqslant C_{1,|\alpha|} \begin{cases} \log\left(\frac{2\mu(z,1/\tau)}{|z-w|}\right), & |z-w| \leqslant \mu\left(z,\frac{1}{\tau}\right),\ |\alpha| = 0, \\ |z-w|^{-|\alpha|}, & |z-w| \leqslant \mu\left(z,\frac{1}{\tau}\right),\ |\alpha| \geqslant 1, \\ \frac{e^{-C_2 \frac{|z-w|}{\mu(z,1/\tau)}} e^{-C_2 \frac{|z-w|}{\mu(w,1/\tau)}}}{\mu(z,1/\tau)^{|\alpha|}}, & |z-w| \geqslant \mu\left(z,\frac{1}{\tau}\right). \end{cases}$$

Also, C_2 does not depend on α.

Near the diagonal, the estimates of Corollary 2 agree with the bounds of $X^\alpha G_p(z,w)$ computed by Christ. Away from the diagonal, the bounds of Christ are governed by a metric equivalent to $d\rho^2 = \frac{1}{\mu(\cdot,1)^2} ds^2$ where ds is the Euclidean metric. Christ shows that for some ϵ, $|X^\alpha G_p(z,w)| \lesssim \mu(z,1)^{-|\alpha|} e^{-\epsilon\rho(z,w)}$, when $|z-w| \geqslant \mu(z,1)$. As shown in Appendix A, in the cases that the author can compute, the two estimates agree. It would be interesting to determine under what circumstances the two estimates agree or disagree.

In $\mathbb{R}^n$, $n \geqslant 3$, Shen [23] obtains estimates for the decay of the fundamental solution of $-\Delta + V$, V is a nonnegative Radon measure. Interestingly, his estimates are sharp even though they are higher dimensional versions of Christ's estimates which are not sharp. This signifies there is additional structure in the special relationship between a and V in the magnetic Schrödinger operator $\Box_{\tau p}$.

Once we have estimates for all $\tau \in \mathbb{R}$, estimates on $H_{\tau p}(s,z,w)$ will have many applications to questions in several complex variables. Pointwise estimates for $H_{\tau p}(s,z,w)$ and its derivatives when $\tau < 0$ is the subject of [20]. A difficulty lies in the fact that techniques from parabolic operator theory and quantum mechanics do not seem to work. In the Schrödinger operator representation, $2\Box_{\tau p} = \frac{1}{2}(-i\nabla - a)^2 + V$, $V \leqslant 0$, and unbounded. Writing $\Box_{\tau p}$ as a parabolic operator, this means the unbounded 0th order term may not be positive. We plan to use our estimates of $H_{\tau p}(s,z,w)$ to prove exponential decay for the heat kernel of [14], an improvement of the rapid decay shown by Nagel and Stein. We also hope to use the OPF operator and heat kernel results to build on the work of [16] by proving pointwise estimates on the heat kernel on the boundary of decoupled domains in $\mathbb{C}^n$, i.e. domains of the form $\Omega = \{(z_1, \dots, z_n)\colon \operatorname{Im} z_n < \sum_{j=1}^{n-1} p_j(z_j)\}$ where p_j are nonharmonic, subharmonic polynomials.

2. Notation and definitions

2.1. Notation for operators on $\mathbb{C}$

For the remainder of the paper, let p be a subharmonic, nonharmonic polynomial. It will be important for us to write p centered around an arbitrary point $z \in \mathbb{C}$, and we set:

$$a^z_{jk} = \frac{1}{j!k!} \frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(z). \tag{3}$$

We need the following functions two "size" functions to write down the size and cancellation conditions for both OPF operators and NIS operators. Let

$$\Lambda(z, \delta) = \sum_{j,k \geqslant 1} \left|a^z_{jk}\right| \delta^{j+k} \tag{4}$$

and

$$\mu(z, \delta) = \inf_{j,k \geqslant 1} \left| \frac{\delta}{a^z_{jk}} \right|^{1/(j+k)}. \tag{5}$$

$\Lambda(z, \delta)$ and $\mu(z, \delta)$ are geometric objects from the Carnot–Carathéodory geometry developed by Nagel, Stein, and Wainger [13,17]. The functions also arise in the analysis of magnetic Schrödinger operators with electric potentials [11,22,23]. It follows $\mu(z, \delta)$ is an approximate inverse to $\Lambda(z, \delta)$. This means that if $\delta > 0$,

$$\mu\bigl(z, \Lambda(z, \delta)\bigr) \sim \delta \quad \text{and} \quad \Lambda\bigl(z, \mu(z, \delta)\bigr) \sim \delta.$$

We use the notation $a \lesssim b$ if $a \leqslant Cb$ where C is a constant that may depend on the dimension 2 and the degree of p. We say that $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

Denote the "twist" at w, centered at z by

$$T(w, z) = -2 \operatorname{Im}\left(\sum_{j \geqslant 1} \frac{1}{j!} \frac{\partial^j p}{\partial z^j}(z)(w - z)^j \right). \tag{6}$$

Also associated to a polynomial p and the parameter $\tau \in \mathbb{R}$ are the weighted differential operators

$$\bar{Z}_{\tau p,z} = \frac{\partial}{\partial \bar{z}} + \tau \frac{\partial p}{\partial \bar{z}} = e^{-\tau p} \frac{\partial p}{\partial \bar{z}} e^{\tau p}, \qquad Z_{\tau p,z} = \frac{\partial}{\partial z} - \tau \frac{\partial p}{\partial z} = e^{\tau p} \frac{\partial p}{\partial z} e^{-\tau p}$$

and

$$\overline{W}_{\tau p,w} = \frac{\partial}{\partial \bar{w}} - \tau \frac{\partial p}{\partial \bar{w}} = e^{\tau p} \frac{\partial p}{\partial \bar{w}} e^{-\tau p}, \qquad W_{\tau p,w} = \frac{\partial}{\partial w} + \tau \frac{\partial p}{\partial w} = e^{-\tau p} \frac{\partial p}{\partial w} e^{\tau p}.$$

$\bar{Z}_{\tau p,z}$ and $Z_{\tau p,z}$ arise naturally as described above. The need for $\overline{W}_{\tau p,w}$ and $W_{\tau p,w}$ is explained below.

We think of τ as fixed and the operators $\bar{Z}_{\tau p,z}$, $Z_{\tau p,z}$, $\overline{W}_{\tau p,w}$, and $W_{\tau p,w}$ as acting on functions defined on $\mathbb{C}$. Also, we will omit the variables z and w from subscripts when the application is unambiguous. Observe that $\overline{(Z_{\tau p})} = \overline{W}_{\tau p}$ and $\overline{(\bar{Z}_{\tau p})} = W_{\tau p}$. We let X_1 and X_2 denote the "real" and "imaginary" parts of $Z_{\tau p}$, that is,

$$X_1 = Z_{\tau p} + \bar{Z}_{\tau p} = \frac{\partial}{\partial x_1} + i\tau \frac{\partial p}{\partial x_2}, \qquad X_2 = i(Z_{\tau p} - \bar{Z}_{\tau p}) = \frac{\partial}{\partial x_2} - i\tau \frac{\partial p}{\partial x_1}.$$

Analogously to X_1 and X_2, define

$$U_1 = W_{\tau p} + \overline{W}_{\tau p} = \frac{\partial}{\partial x_1} - i\tau \frac{\partial p}{\partial x_2}, \qquad U_2 = i(W_{\tau p} - \overline{W}_{\tau p}) = \frac{\partial}{\partial x_2} + i\tau \frac{\partial p}{\partial x_1}.$$

We need to establish notation for adjoints. If T is an operator (either bounded or closed and densely defined) on a Hilbert space with inner product $(\cdot,\cdot)$, let T^* be the Hilbert space adjoint of T. This means that if $f \in \operatorname{Dom} T$ and $g \in \operatorname{Dom} T^*$, then $(Tf, g) = (f, T^* g)$. If U is an unbounded domain in some Euclidean space and T is an operator acting on $C_c^\infty(U)$ or $\mathcal{S}(U) = \{\varphi \in C^\infty(U)\colon \varphi \text{ has rapid decay}\}$, then we denote $T^\#$ as the adjoint in the sense of distributions. This means if K is a distribution or a Schwartz distribution, then $\langle T^\# K, \varphi\rangle = \langle K, T\varphi\rangle$. Note that if T is not $\mathbb{R}$-valued, $T^* \neq T^\#$. It follows easily that

$$\bar{Z}_{\tau p}^\# = -\overline{W}_{\tau p} \quad \text{and} \quad Z_{\tau p}^\# = -W_{\tau p}.$$

Finally, let $M_{\tau p} = e^{i\tau T(w,z)} \frac{\partial}{\partial \tau} e^{-i\tau T(w,z)}$.

2.2. *Definition of OPF operators*

We use the definition from [19]. We say that T_τ is a *one-parameter family* (*OPF*) *of operators* of order m with respect to the polynomial p if the following conditions hold:

(a) There is a function $K_\tau \in C^\infty(((\mathbb{C} \times \mathbb{C}) \setminus \{z = w\}) \times (\mathbb{R} \setminus \{0\}))$ so that for fixed τ, K_τ is a distributional kernel, i.e. if $\varphi, \psi \in C_c^\infty(\mathbb{C})$ and $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$, then $T_\tau[\varphi] \in (C_c^\infty)'(\mathbb{C})$ and

$$\langle T_\tau[\varphi](\cdot), \psi\rangle_{\mathbb{C}} = \iint_{\mathbb{C}\times\mathbb{C}} K_\tau(z, w)\varphi(w)\psi(z)\, dw\, dz.$$

(b) There exists a family of functions $K_{\tau,\epsilon}(z, w) \in C^\infty(\mathbb{C} \times \mathbb{C} \times \mathbb{R})$ so that if $\varphi \in C_c^\infty(\mathbb{C} \times \mathbb{R})$,

$$K_{\tau,\epsilon}[\varphi]_{\mathbb{C}\times\mathbb{R}}(z, \tau) = \int_{\mathbb{C}\times\mathbb{R}} \varphi(w, \tau) K_{\tau,\epsilon}(z, w)\, dw\, d\tau$$

and $\lim_{\epsilon\to 0} K_{\tau,\epsilon}[\varphi]_{\mathbb{C}\times\mathbb{R}}(z) = K_\tau[\varphi]_{\mathbb{C}\times\mathbb{R}}(z)$ in $(C_c^\infty)'(\mathbb{C} \times \mathbb{R})$.
All of the additional conditions are assumed to apply to the kernels $K_{\tau,\epsilon}(z, w)$ uniformly in ϵ.

(c) *Size estimates*. If $Y^J_{\tau p}$ is a product of $|J|$ operators of the form $Y^j_{\tau p} = Z_{\tau p,z}$, $\bar{Z}_{\tau p,z}$, $W_{\tau p,w}$, $\overline{W}_{\tau p,w}$, or $M_{\tau p}$ where $|J| = \ell + n$ and $n = \#\{j\colon\ Y^j_{\tau p} = M_{\tau p}\}$, for any $k \geqslant 0$ there exists a constant $C_{\ell,n,k}$ so that

$$\left|Y^J_{\tau p} K_{\tau,\epsilon}(z,w)\right| \leqslant \frac{C_{\ell,n,k}}{|\tau|^n} \frac{|z-w|^{m-2-\ell}}{|\tau|^k \Lambda(z,|w-z|)^k} \quad \text{if } \begin{cases} m<2, \\ m=2,\ k\geqslant 1, \\ m=2,\ |w-z| > \mu(z,\frac{1}{\tau}). \end{cases} \tag{7}$$

Also, if $m = 2$ and $|w-z| \leqslant \mu(z, \frac{1}{\tau})$, then

$$\left|M^n_{\tau p} K_{\tau,\epsilon}(z,w)\right| \leqslant C_n \begin{cases} \log(\frac{2\mu(z,1/\tau)}{|w-z|}), & n=0, \\ |\tau|^{-n}, & n\geqslant 1. \end{cases} \tag{8}$$

(d) *Cancellation in w*. If $Y^J_{\tau p}$ is a product of $|J|$ operators of the form $Y^j_{\tau p} = Z_{\tau p,z}$, $\bar{Z}_{\tau p,z}$, $W_{\tau p,w}$, $\overline{W}_{\tau p,w}$, or $M_{\tau p}$ where $|J| = \ell + n$ and $n = \#\{j\colon\ Y^j_{\tau p} = M_{\tau p}\}$, for any $k \geqslant 0$ there exists a constant $C_{\ell,n,k}$ and N_ℓ so that for $\varphi \in C^\infty_c(D(z_0,\delta))$,

$$\sup_{z\in\mathbb{C}} \left| \int_{\mathbb{C}} Y^J_{\tau p} K_{\tau,\epsilon}(z,w)\varphi(w)\,dw \right|$$
$$\leqslant \frac{C_{\ell,n,k}}{|\tau|^n} \begin{cases} \delta^2\big(\log(\frac{2\mu(z,1/\tau)}{\delta})\|\varphi\|_{L^\infty} + \sum_{1\leqslant|I|\leqslant N_0} \delta^{|I|}\|X^I_{\tau p}\varphi(w)\|_{L^\infty}\big) \\ \qquad \delta<\mu(z,\frac{1}{\tau}) \text{ and } m=2,\ \ell=0, \\ \frac{\delta^{m-\ell}}{|\tau|^k \Lambda(z,\delta)^k} \sum_{|I|\leqslant N_\ell} \delta^{|I|}\|X^I_{\tau p}\varphi\|_{L^\infty(\mathbb{C})} \quad \text{otherwise,} \end{cases} \tag{9}$$

where $X^I_{\tau p}$ is composed solely of $Z_{\tau p}$ and $\bar{Z}_{\tau p}$.

(e) *Cancellation in τ*. If $X^J_{\tau p}$ is a product of $|J|$ operators of the form $X^j_{\tau p} = Z_{\tau p,z}$, $\bar{Z}_{\tau p,z}$ or $W_{\tau p,w}$, $\overline{W}_{\tau p,w}$ and $|J| = n$, there exists a constant C_n so that

$$\int_{\mathbb{R}} X^J_{\tau p}\big(e^{i\tau t} K_{\tau,\epsilon}(z,w)\big)\,d\tau \leqslant C_n \frac{\mu(z,t+T(w,z))^{m-n}}{\mu(z,t+T(w,z))^2 |t+T(w,z)|}. \tag{10}$$

(f) *Adjoint*. Properties (a)–(e) also hold for the adjoint operator T^*_τ whose distribution kernel is given by $\overline{K_{\tau,\epsilon}(w,z)}$.

The following results from [19] are essential tools in the proof of Theorem 1.

Theorem 3. *If T_τ is an OPF operator of order* 0*, then T_τ, T^*_τ are bounded operators from $L^q(\mathbb{C})$ to $L^q(\mathbb{C})$, $1<q<\infty$, with a constant independent of τ but depending on q.*

Theorem 4. *Given a subharmonic, nonharmonic polynomial $p:\mathbb{C}\to\mathbb{R}$, there is a one-to-one correspondence between OPF operators of order $m\leqslant 2$ with respect to p and NIS operators of order $m\leqslant 2$ on the polynomial model $M_p = \{(z_1,z_2)\in\mathbb{C}^2\colon\ \operatorname{Im} z_2 = p(z_1)\}$. The correspondence is given by a partial Fourier transform in* $\operatorname{Re} z_2$.

There are multiple definitions of NIS operators (e.g. [14,18]). This equivalence is with the definition in [18].

3. The heat equation and smoothness of the heat kernel

For the remainder of the work, we will primarily be concerned with inverting the "Laplace" operator

$$\Box_{\tau p} = -\bar{Z}_{\tau p} Z_{\tau p}$$

via the heat semigroup $e^{-s\,\Box_{\tau p}}$. We assume that $\tau > 0$ and define the heat operator

$$\mathcal{H}_{\tau p} = \frac{\partial}{\partial s} + \Box_{\tau p}.$$

Given a function f defined on $\mathbb{C}$, we study the initial value problem of finding smooth $u : (0,\infty) \times \mathbb{C} \to \mathbb{C}$ so that

$$\begin{cases} \mathcal{H}_{\tau p}[u](s,z) = 0, & s > 0,\ z \in \mathbb{C}, \\ \lim_{s\to 0} u(s,\cdot) = f(\cdot) & \text{with convergence in an appropriate norm.} \end{cases} \tag{11}$$

Let α be a multi-index. We let X^α be a product of $|\alpha|$ operators of the form $X = X_1$ or X_2. Similarly, U^α is a product of $|\alpha|$ operators of the form $U = U_1$ or U_2.

4. The heat semigroup $e^{-s\,\Box_{\tau p}}$ on $L^2(\mathbb{C})$

We know that $\bar{Z}_{\tau p}$ and $Z_{\tau p}$ are closed, densely defined operators on $L^2(\mathbb{C})$. As in Nagel and Stein [14], the spectral theorem for unbounded operators (see [21]) proves:

Theorem 5. *$\Box_{\tau p}$ is the infinitesimal generator of $e^{-s\,\Box_{\tau p}}$, a strongly continuous semigroup of bounded operators on $L^2(\mathbb{C})$ for $s > 0$. For $f \in L^2(\mathbb{C})$, the following hold*:

(a) $\lim_{s\to 0} \|e^{-s\,\Box_{\tau p}} f - f\|_{L^2(\mathbb{C})} = 0$;
(b) *For $s > 0$, these operators are contractions, that is,*

$$\left\| e^{-s\,\Box_{\tau p}} f \right\|_{L^2(\mathbb{C})} \leqslant \|f\|_{L^2(\mathbb{C})};$$

(c) *For $f \in \mathrm{Dom}(\Box_{\tau p})$,*

$$\left\| e^{-s\,\Box_{\tau p}} f - f \right\|_{L^2(\mathbb{C})} \leqslant s \|\Box_{\tau p} f\|_{L^2(\mathbb{C})};$$

(d) *For $s > 0$ and all j, $\mathrm{Range}(e^{-s\,\Box_{\tau p}}) \subset \mathrm{Dom}(\Box_{\tau p}^j)$. Also, $\Box_{\tau p}^j e^{-s\,\Box_{\tau p}}$ is a bounded operator on $L^2(\mathbb{C})$ with*

$$\left\| \Box_{\tau p}^j e^{-s\,\Box_{\tau p}} f \right\|_{L^2(\mathbb{C})} \leqslant \left(\frac{j}{e}\right) s^{-j} \|f\|_{L^2(\mathbb{C})};$$

(e) *For any* $f \in L^2(\mathbb{C})$ *and* $s > 0$, *the Hilbert space-valued function* $u(s) = e^{-s\,\square_{\tau p}} f$ *satisfies*

$$\left(\frac{\partial}{\partial s} + \square_{\tau p}\right) u(s) = 0.$$

5. Regularity of the heat kernel

For each $s > 0$, define the bounded operator $H^s_{\tau p} : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ by

$$H^s_{\tau p}[f] = e^{-s\,\square_{\tau p}}[f].$$

Our main result on the existence and regularity of $H_{\tau p}(s, z, w)$ is the following theorem.

Theorem 6. *Fix* $\tau > 0$. *There is a function* $H_{\tau p} \in C^\infty((0, \infty) \times \mathbb{C} \times \mathbb{C})$ *so that for all* $f \in L^2(\mathbb{C})$,

$$H^s_{\tau p}[f](z) = \int_{\mathbb{C}} H_{\tau p}(s, z, w) f(w)\, dw. \tag{12}$$

Moreover, for each fixed $s > 0$ *and* $z \in \mathbb{C}$, *the function* $w \mapsto H_{\tau p}(s, z, w)$ *is in* $L^2(\mathbb{C})$, *so the integral defined in* (12) *converges absolutely. Also,*

(a) $H_{\tau p}(s, z, w) = \overline{H_{\tau p}(s, w, z)}$;
(b) *For* $(s, z, w) \in (0, \infty) \times \mathbb{C} \times \mathbb{C}$,

$$\left(\frac{\partial}{\partial s} + \square_{\tau p,z}\right)[H_{\tau p}](s, z, w) = \left(\frac{\partial}{\partial s} + \square^{\#}_{\tau p,w}\right)[H_{\tau p}](s, z, w) = 0;$$

(c) *For any integers* $j, k \geqslant 0$,

$$\square^j_{\tau p,z}\left(\square^{\#}_{\tau p,w}\right)^k H_{\tau p}(s, z, w) = \square^{j+k}_{\tau p,z} H_{\tau p}(s, z, w) = \left(\square^{\#}_{\tau p,w}\right)^{j+k} H_{\tau p}(s, z, w);$$

(d) *For all integers* j *and multi-indices* α, β, *the functions*

$$w \mapsto \frac{\partial^j}{\partial s^j} X^\alpha_z U^\beta_w H_{\tau p}(s, z, w)$$

are in $L^2(\mathbb{C})$ *and there is a constant* $c_{j,\alpha,\beta}$ *so that for* $R < R_{\tau p}(z)$,

$$\left\| \frac{\partial^j}{\partial s^j} X^\alpha_z U^\beta_w H_{\tau p}(s, z, \cdot) \right\|_{L^2(\mathbb{C})} \leqslant \frac{C_{\alpha,\beta,j}}{R} s^{-\frac{\alpha+\beta}{2}-j}\left(1 + s^{-1}\right);$$

(e) *The conclusions of* (d) *hold with the roles of* z *and* w *interchanged.*

6. Properties of OPF operators

To prove Theorem 6, we need to establish properties of OPF operators. We follow the line of argument for NIS operators in [14,18]. Since we are working with a fixed polynomial p, we omit τp from subscripts when the application is clear.

Lemma 7. *Let A_τ and B_τ be order 0 OPF operators, and let $X = X_1$ or X_2. There exist order 0 OPF operators A_1, A_2, B_1 and B_2 so that*

$$XA_\tau = A_1X_1 + A_2X_2, \qquad B_\tau X = X_1B_1 + X_2B_2.$$

Proof. We know from results for NIS operators and Theorem 4 that $X_1^2 + X_2^2$ is invertible with an inverse K_τ that is an OPF operator smoothing of order 2. Thus, we can write

$$XA_\tau = (XA_\tau K_\tau X_1)X_1 + (XA_\tau K_\tau X_2)X_2 = A_1X_1 + A_2X_2.$$

A similar argument proves the result for B_τ. □

Corollary 8. *Let A_τ and B_τ be order 0 OPF operators and α a multi-index where $|\alpha| = k \geqslant 1$. There exist finite sets I and J of multi-indices α_i, $|\alpha_i| = k$, and β_j, $|\beta_j| = k$, respectively, so that*

$$X^\alpha A_\tau = \sum_{\alpha_i \in I} A_i X^{\alpha_i}, \qquad B_\tau X^\alpha = \sum_{\alpha_j \in J} X^\beta B_j$$

for some order 0 OPF operators A_i and B_j.

Proof. Induction. □

Let $[n]$ denote the greatest integer less than or equal to n. The proofs of the following two propositions are in [14].

Proposition 9. *Let α be a multi-index.*

(a) *If $|\alpha|$ is even, there exists an order 0 OPF operator A_τ so that*

$$X^\alpha = \Box_{\tau p}^{|\alpha|/2} A_\tau.$$

(b) *If $|\alpha|$ is odd, there exist order 0 OPF operators A_1 and A_2 so that*

$$X^\alpha = \Box_{\tau p}^{[|\alpha|/2]}(X_1A_1 + X_2A_2).$$

Proposition 10. *Let α be a multi-index.*

(a) *If $|\alpha|$ is even, there exists an order 0 OPF operator B_τ so that*

$$X^\alpha = B_\tau \Box_{\tau p}^{|\alpha|/2};$$

(b) *If* $|\alpha|$ *is odd, there exist order* 0 *OPF operators* B_1 *and* B_2 *so that*

$$X^\alpha = \Box_{\tau p}^{[|\alpha|/2]}(B_1 X_1 + B_2 X_2);$$

(c) *Alternatively, if* $|\alpha|$ *is odd and* $X^\alpha = X^\beta X$ *where* $X = X_1$ *or* X_2*, then there exists an order* 0 *OPF operator* B_τ *so that*

$$X^\alpha = B_\tau \Box_{\tau p}^{|\beta|/2} X.$$

Proposition 11. *Let* $X = X_1$ *or* X_2*. There is a constant* C *so that if* $\varphi \in C_c^\infty(\mathbb{C})$*, then for all* $r > 0$

$$\|X[\varphi]\|_{L^2(\mathbb{C})} \leqslant C\big(r\|\Box_{\tau p}\varphi\|_{L^2(\mathbb{C})} + r^{-1}\|\varphi\|_{L^2(\mathbb{C})}\big).$$

Proof. First, note that $X^* = -X$. Using Proposition 10, we compute

$$\begin{aligned}\|X[\varphi]\|_{L^2(\mathbb{C})}^2 &= \big(X[\varphi], X[\varphi]\big) = -\big(X^2[\varphi], \varphi\big) \leqslant \big|\big(A_\tau \Box_{\tau p}[\varphi], \varphi\big)\big| \\ &\leqslant C\big(r^2\|\Box_{\tau p}\varphi\|_{L^2(\mathbb{C})}^2 + r^{-2}\|\varphi\|_{L^2(\mathbb{C})}^2\big). \qquad \Box\end{aligned}$$

Corollary 12. *Let* α *be a multi-index. There exists a constant* $C_{|\alpha|}$ *so that if* $\varphi \in C_c^\infty(\mathbb{C})$*, then*

$$\|X^\alpha[\varphi]\|_{L^2(\mathbb{C})} \leqslant C_\alpha \sum_{j=0}^{[|\alpha|/2]+1} \|\Box_{\tau p}^j[\varphi]\|_{L^2(\mathbb{C})}.$$

Proof. Proof by induction. The base case is Proposition 11 and the inductive step is a repetition of the argument in the proof of Proposition 11. $\Box$

We now prove the Sobolev type theorem.

Theorem 13. *Let*

$$R_{\tau p}(z) = \inf_{j,k\geqslant 0} \frac{1}{|\tau a_{jk}^z|^{1/(j+k)}}.$$

There is a constant $C > 0$ *so that if* $f \in C^\infty(\mathbb{C})$*,* $z \in \mathbb{C}$ *and* $0 < R < R_{\tau p}(z)$*,*

$$\sup_{D(z,R)} |f| \leqslant \frac{C}{R} \sum_{|\alpha|\leqslant 2} R^{|\alpha|}\|X^\alpha f\|_{L^2(D(z,2R))}.$$

Also, if $f \in C^\infty(\mathbb{C}) \cap L^2(\mathbb{C})$*, then*

$$\sup_{D(z,R)} |f| \leqslant \frac{C}{R}\big(\|f\|_{L^2(D(z,2R))} + R^2\|\Box_{\tau p} f\|_{L^2(D(z,2R))}\big).$$

Proof. Let $f \in C^\infty(\mathbb{C})$ and $z \in \mathbb{C}$. An application of Plancherel's theorem shows

$$\sup_{D(z_0,R)} |f(z)| \leqslant \frac{C}{R} \sum_{|\alpha| \leqslant 2} R^{|\alpha|} \|D^\alpha f(z)\|_{L^2(D(z_0,2R))}. \tag{13}$$

To pass from ordinary derivatives to products of $Z_{\tau p}$ and $\bar{Z}_{\tau p}$, first observe that if $|w-z| < R < R_{\tau p}(z)$, then

$$\begin{aligned}\left|\tau \frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(w)\right| &= \left|\tau \sum_{\substack{j' \geqslant j \\ k' \geqslant k}} \frac{1}{(j-j')!(k-k')!} \frac{\partial^{j'+k'} p}{\partial z^{j'} \partial \bar{z}^{k'}}(z)(w-z)^{j'-j}\, \overline{(w-z)}^{k'-k}\right| \\ &\leqslant \frac{C\tau}{R^{j+k}} \sum_{\substack{j' \geqslant j \\ k' \geqslant k}} \left|\frac{\partial^{j'+k'} p}{\partial z^{j'} \partial \bar{z}^{k'}}(z)\right| R^{j'+k'} \leqslant \frac{C}{R^{j+k}},\end{aligned}$$

where C does not depend on p or R. The proof of the first part of the theorem now follows easily since

$$\frac{\partial f}{\partial \bar{z}}(w) = \bar{Z}_{\tau p} f(w) - \tau \frac{\partial p}{\partial \bar{z}}(w) f(w).$$

This means

$$\left|\frac{\partial f}{\partial \bar{z}}(w)\right| \leqslant |\bar{Z}_{\tau p} f(w)| + \frac{C}{R}|f(w)|$$

and similarly for $|\frac{\partial f}{\partial z}(w)|$. The estimates for the second derivatives proceed in the same fashion. For example,

$$\begin{aligned}\left|\frac{\partial^2 f}{\partial z \partial \bar{z}}(w)\right| &= \left|\bar{Z}_{\tau p} Z_{\tau p}[f](w) - \tau \frac{\partial p(w)}{\partial \bar{z}} \frac{\partial f(w)}{\partial z} + \tau^2 \frac{\partial p(w)}{\partial \bar{z}} \frac{\partial p(w)}{\partial z} f(w)\right. \\ &\quad \left. + \tau \frac{\partial p(w)}{\partial z} \frac{\partial f(w)}{\partial \bar{z}} + \tau \frac{\partial^2 p(w)}{\partial z \partial \bar{z}} f(w)\right| \\ &\leqslant |\bar{Z}_{\tau p} Z_{\tau p}[f](w)| + \frac{C}{R}|\nabla f(w)| + \frac{C}{R^2}|f|.\end{aligned}$$

The other second derivatives of f are handled similarly. Thus, every term in (13) is well controlled by $Z_{\tau p}$ and $\bar{Z}_{\tau p}$ derivatives. The proof of the latter part of the theorem follows from Propositions 10 and 11.

Remark 14. In Theorem 13, if $\frac{\partial^j p}{\partial z^j}(z) = \frac{\partial^j p}{\partial \bar{z}^j}(z) = 0$ for $j = 0, 1, \ldots, \deg(p)$, then $R_{\tau p}(z) = \mu(z, 1/\tau)$, a fact which will be useful later.

7. Proof of Theorem 6

To prove Theorem 6, we need some a priori estimates.

Lemma 15. *There are constants $C_{\alpha,\beta}$ so that for any multi-indices α and β, any $s > 0$, and $\varphi \in C_c^\infty(\mathbb{C})$,*

$$\left\| X^\alpha H_{\tau p}^s\left[X^\beta \varphi\right]\right\|_{L^2(\mathbb{C})} \leqslant C_{\alpha,\beta} s^{-\frac{|\alpha|+|\beta|}{2}} \|\varphi\|_{L^2(\mathbb{C})}.$$

Proof. We first assume that $|\alpha|$ and $|\beta|$ are even. From Proposition 9, there exists an order 0 OPF operator A_τ so that

$$H_{\tau p}^s\left[X^\beta \varphi\right] = H_{\tau p}^s \,\Box_{\tau p}^{|\beta|/2} A_\tau \varphi.$$

Hence, by Proposition 10 and Theorem 5(d) we have an order zero family B_τ so that

$$\begin{aligned}\left\| X^\alpha H_{\tau p}^s\left[X^\beta \varphi\right]\right\|_{L^2(\mathbb{C})} &= \left\| X^\alpha \Box_{\tau p}^{|\beta|/2} H_{\tau p}^s[A_\tau \varphi]\right\|_{L^2(\mathbb{C})} = \left\| B_\tau \Box_{\tau p}^{|\alpha|/2} \Box_{\tau p}^{|\beta|/2} H_{\tau p}^s[A_\tau\varphi]\right\|_{L^2(\mathbb{C})}\\ &\leqslant C_{\alpha,\beta} s^{-\frac{|\alpha|+|\beta|}{2}} \|\varphi\|_{L^2(\mathbb{C})}.\end{aligned}$$

The $|\alpha|$ and $|\beta|$ odd cases follow easily from the even case, an application of Propositions 9 and 10, and the following two arguments. One, if X is either X_1 or X_2 then from Proposition 11 with $r = s^{1/2}$,

$$\left\| X H_{\tau p}^s \varphi\right\|_{L^2(\mathbb{C})} \leqslant C\left(s^{\frac12}\left\|\Box_{\tau p} H_{\tau p}^s\varphi\right\|_{L^2(\mathbb{C})} + s^{-\frac12}\left\|H_{\tau p}^s\varphi\right\|_{L^2(\mathbb{C})}\right) \leqslant C s^{-\frac12}\|\varphi\|_{L^2(\mathbb{C})}.$$

Two, since $X^* = -X$, applying the previous inequality to $H_{\tau p}^s X\varphi$, we have

$$\begin{aligned}\left\| H_{\tau p}^s X\varphi\right\|_{L^2(\mathbb{C})}^2 &= \left(H_{\tau p}^s X\varphi, H_{\tau p}^s X\varphi\right) = -\left(\varphi, X H_{\tau p}^s H_{\tau p}^s X\varphi\right)\\ &\leqslant \|\varphi\|_{L^2(\mathbb{C})}\left\| X H_{\tau p}^s H_{\tau p}^s X\varphi\right\|_{L^2(\mathbb{C})} \leqslant C s^{-\frac12}\|\varphi\|_{L^2(\mathbb{C})}\left\|H_{\tau p}^s X\varphi\right\|_{L^2(\mathbb{C})}. \qquad \Box\end{aligned}$$

Lemma 16. *For $s > 0$ and $f \in L^2(\mathbb{C})$, $H_{\tau p}^s[f]$ is $C^\infty(\mathbb{C})$. Given a multi-index γ, there is a constant $C_{|\gamma|}$ so that for $z \in \mathbb{C}$ and $R < \min\{R_{\tau p}(z), 1\}$ where $R_{\tau p}(z)$ is the constant from Theorem 13,*

$$\left|X^\gamma H_{\tau p}^s[f](z)\right| \leqslant C_{|\gamma|} R^{-1} s^{-\frac{|\gamma|}{2}}\left(1 + s^{-1}\right)\|f\|_{L^2(\mathbb{C})}.$$

Proof. We can find $\varphi_n \in C_c^\infty(\mathbb{C})$ so that $\varphi_n \to f$ in $L^2(\mathbb{C})$. It follows immediately from Lemma 15 that $X^\gamma H_{\tau p}^s[f] \in L^2(\mathbb{C})$ and

$$X^\gamma H_{\tau p}^s[\varphi_n] \to X^\gamma H_{\tau p}^s[f]$$

in $L^2(\mathbb{C})$, hence

$$\left\|X^\gamma H_{\tau p}^s[f]\right\|_{L^2(\mathbb{C})} \leqslant C_{|\gamma|} R^{-1} s^{-\frac{|\gamma|}{2}}\|f\|_{L^2(\mathbb{C})}.$$

From these inequalities, we can show that all $Z_{\tau p}$ and $\bar{Z}_{\tau p}$ derivatives of $H^s_{\tau p}[f]$ are in $L^2(\mathbb{C})$. To pass from L^2-bounds of $Z_{\tau p}$ and $\bar{Z}_{\tau p}$ derivatives to a local L^2-bound for ordinary derivatives, we can use the argument of Theorem 13. Thus, $H^s_{\tau p}[f]$ is $C^\infty(\mathbb{C})$, and by Theorem 13,

$$\sup_{D(z,R)} \left|X^\gamma H^s_{\tau p}[f]\right| \leqslant \frac{C}{R} \sum_{|\alpha|\leqslant 2} R^{|\alpha|} \left\|X^\alpha X^\gamma H^s_{\tau p}[f]\right\|_{L^2(\mathbb{C})} \leqslant \frac{C}{R} \sum_{|\alpha|\leqslant 2} s^{-\frac{|\alpha|+|\gamma|}{2}} \|f\|_{L^2(\mathbb{C})}$$
$$\leqslant C_{|\gamma|} R^{-1} s^{-\frac{|\gamma|}{2}} \left(1+s^{-1}\right) \|f\|_{L^2(\mathbb{C})}. \qquad \square$$

Recall the following standard fact.

Lemma 17. *If $x_1 \mapsto \partial^\alpha_{x_1} f(x_1, x_2)$ and $x_2 \mapsto \partial^\alpha_{x_2} f(x_1, x_2)$ are in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ for all multi-indices α, then $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.*

We are now ready to prove Theorem 6. We follow the line of argument in [14] and show the proof for completeness.

Proof of Theorem 6. For every multi-index α, Lemma 16 and Theorem 13 show that the functional on $L^2(\mathbb{C})$ defined by

$$f \mapsto \partial^\alpha H^s_{\tau p}[f]$$

is bounded. By the Riesz Representation Theorem, a consequence of these facts is the existence of functions $H^{\alpha,s,z}_{\tau p}(w)$ so that

$$\partial^\alpha H^s_{\tau p}[f](z) = \int_{\mathbb{C}} H^{\alpha,s,z}_{\tau p}(w) f(w)\, dw.$$

Define $H^\alpha_{\tau p}(s,z,w) = H^{\alpha,s,z}_{\tau p}(w)$. Also set $H_{\tau p}(s,z,w) = H^0_{\tau p}(s,z,w)$. Then $H^\alpha_{\tau p}$ is a function on $(0,\infty) \times \mathbb{C} \times \mathbb{C}$ with the property that $w \mapsto H^\alpha_{\tau p}(s,z,w)$ is in $L^2(\mathbb{C})$. Thus, we have

$$H^s_{\tau p}[f](z) = \int_{\mathbb{C}} H_{\tau p}(s,z,w) f(w)\, dw$$

and for every derivative ∂^α_z,

$$\partial^\alpha_z \left(\int_{\mathbb{C}} H_{\tau p}(s,z,w) f(w)\, dw \right) = \int_{\mathbb{C}} H^\alpha_{\tau p}(s,z,w) f(w)\, dw.$$

We will show that $\partial^\alpha_z H_{\tau p}(s,z,w) = H^\alpha_{\tau p}(s,z,w)$. Let $\varphi, \psi \in C^\infty(\mathbb{C})$. By the Schwartz Kernel Theorem,

$$\left\langle \partial^\alpha_z H^s_{\tau p}[\psi], \varphi \right\rangle_{\mathbb{C}} = \left\langle (-1)^{|\alpha|} H^s_{\tau p}[\psi], \partial^\alpha_z \varphi \right\rangle_{\mathbb{C}} = \left\langle (-1)^{|\alpha|} H^s_{\tau p}, \psi \otimes \partial^\alpha_z \varphi \right\rangle_{\mathbb{C}\times\mathbb{C}}$$
$$= \left\langle \partial^\alpha_z H^s_{\tau p}, \psi \otimes \varphi \right\rangle_{\mathbb{C}\times\mathbb{C}} = \left\langle \left(\partial^\alpha_z H^s_{\tau p}\right)[\psi], \varphi \right\rangle_{\mathbb{C}}.$$

Thus, we have shown that

$$\partial_z^\alpha H_{\tau p}(s,z,w) = H_{\tau p}^\alpha(s,z,w) \tag{14}$$

in $\mathcal{D}'(\mathbb{C})$ and that for each α, $w \mapsto \partial_z^\alpha H_{\tau p}(s,z,w)$ is a function in $L^2(\mathbb{C})$.

Next, we know that $H_{\tau p}^s$ is self-adjoint, so

$$\int_{\mathbb{C}} H_{\tau p}^s[\psi](z)\overline{\varphi(z)}\,dz = \int_{\mathbb{C}} \psi(w)\overline{H_{\tau p}^s[\varphi](w)}\,dw.$$

As an immediate consequence of this equality and (7), we have

$$\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s,z,w)\psi(w)\overline{\varphi(z)}\,dw\,dz = \int_{\mathbb{C}}\int_{\mathbb{C}} \overline{H_{\tau p}(s,w,z)}\psi(w)\overline{\varphi(z)}\,dz\,dw.$$

It follows that $H_{\tau p}(s,z,w) = \overline{H_{\tau p}(s,w,z)}$, conclusion (a).

As a consequence of (a) and the fact that $w \mapsto H_{\tau p}^s(s,z,w)$ belongs to $L^2(\mathbb{C})$, $z \mapsto H_{\tau p}^s(s,z,w)$ belongs to $L^2(\mathbb{C})$. By (14), it follows that every z derivative also belongs to L^2. Thus by Lemma 17, $H_{\tau p}(s,z,w)$ is $C^\infty(\mathbb{C}\times\mathbb{C})$ for fixed $s>0$.

We know $\Box_{\tau p}^j H_{\tau p}^s = H_{\tau p}^s \Box_{\tau p}^j$. The implication of the self-adjointness of $\Box_{\tau p}$ is that on the kernel side,

$$\Box_{\tau p,z}^j H_{\tau p}(s,z,w) = \left(\Box_{\tau p,w}^\#\right)^j H_{\tau p}(s,z,w).$$

From this, (c) follows quickly because $\Box_{\tau p,z}^{j+k} H_{\tau p}^s = \Box_{\tau p,z}^j H_{\tau p}^s \Box_{\tau p,w}^k$.

Next, by Theorem 5(e),

$$\left(\frac{\partial}{\partial s} + \Box_{\tau p}\right)\left[H_{\tau p}^s[f]\right](z) = 0.$$

Fixing $z\in\mathbb{C}$, integration against test functions in $(0,\infty)\times\mathbb{C}$ shows that in $\mathcal{D}'((0,\infty)\times\mathbb{C})$,

$$\begin{aligned}
0 &= \left\langle \left(\frac{\partial}{\partial s} + \Box_{\tau p}\right)\left[H_{\tau p}^s[f]\right](z), \varphi\right\rangle = \iint_{(0,\infty)\times\mathbb{C}} H_{\tau p}(s,z,w)\left(-\frac{\partial}{\partial s} + \Box_{\tau p,z}^\#\right)\varphi(s,z) f(w)\,dw\,ds \\
&= \iint_{(0,\infty)\times\mathbb{C}} \left(\frac{\partial}{\partial s} + \Box_{\tau p,z}\right) H_{\tau p}(s,z,w)\varphi(s,z) f(w)\,dw\,ds.
\end{aligned}$$

Thus,

$$\frac{\partial}{\partial s} H_{\tau p}(s,z,w) = -\Box_{\tau p,z} H_{\tau p}(s,z,w).$$

Then we have

$$\frac{\partial^2}{\partial s^2} H_{\tau p}(s,z,w) = -\frac{\partial}{\partial s} \Box_{\tau p,z} H_{\tau p}(s,z,w)$$
$$= -\Box_{\tau p,z} \frac{\partial}{\partial s} H_{\tau p}(s,z,w) = \Box^2_{\tau p,z} H_{\tau p}(s,z,w).$$

Iterating this argument shows

$$\frac{\partial^j}{\partial s^j} H_{\tau p}(s,z,w) = (-1)^j \Box^j_{\tau p,z} H_{\tau p}(s,z,w). \tag{15}$$

We know, however, that for each j, $\Box^j_{\tau p,z} H_{\tau p}(s,z,w) \in L^2_{\mathrm{loc}}((0,\infty)\times\mathbb{C}\times\mathbb{C})$. As before, this is enough to show $H_{\tau p} \in C^\infty((0,\infty)\times\mathbb{C}\times\mathbb{C})$. In particular, (15) hold in the classical sense, so (b) is proved.

For α, β, and j, Lemma 15 shows that there is a constant $C_{\alpha,\beta,j}$ so that for $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{C})$,

$$\big\| X^\alpha \Box^j_{\tau p} H^s_{\tau p}\big[X^\beta[\varphi]\big]\big\|_{L^2(\mathbb{C})} \leqslant C_{\alpha,\beta} s^{-\frac{|\alpha|+|\beta|}{2}-j} \|\varphi\|_{L^2(\mathbb{C})}.$$

Then by Theorem 13, for $R < R_{\tau p}(z)$,

$$\sup_{D(z,R)} \big|X^\alpha \Box^j_{\tau p} H^s_{\tau p}\big[X^\beta[\varphi]\big]\big| \leqslant \frac{C}{R} \sum_{|\gamma|\leqslant 2} R^{|\gamma|} \big\| X^{\alpha+\gamma} \Box^j_{\tau p} H^s_{\tau p}\big[X^\beta[\varphi]\big]\big\|_{L^2(\mathbb{C})}$$
$$\leqslant \frac{C}{R} s^{-\frac{\alpha+\beta}{2}-j}\big(1+s^{-1}\big)\|\varphi\|_{L^2(\mathbb{C})}.$$

Also, since $H_{\tau p}(s,z,w) \in C^\infty((0,\infty)\times\mathbb{C}\times\mathbb{C})$,

$$X^\alpha \Box^j_{\tau p} H^s_{\tau p}\big[X^\beta\varphi\big](z) = \int_{\mathbb{C}} X^\alpha_z \Box^j_{\tau p} H_{\tau p}(s,z,w) X^\beta_w \varphi(w)\, dw$$
$$= (-1)^{j+|\beta|} \int_{\mathbb{C}} \frac{\partial^j}{\partial s^j} X^\alpha_z U^\beta_w H_{\tau p}(s,z,w)\varphi(w)\, dw.$$

From the reverse Hölder inequality and our previous estimate,

$$\left(\int_{\mathbb{C}} \left|\frac{\partial^j}{\partial s^j} X^\alpha_z U^\beta_w H_{\tau p}(s,z,w)\right|^2 dw\right)^{1/2} \leqslant \frac{C_{\alpha,\beta,j}}{R} s^{-\frac{\alpha+\beta}{2}-j}\big(1+s^{-1}\big).$$

This is (d) of the theorem. From (a), we can interchange the roles of z and w to prove (e). This proves the theorem. $\Box$

8. A fundamental solution for $\mathcal{H}_{\tau p}$ on $\mathbb{R} \times \mathbb{C}$

Define the distribution $H^z_{\tau p}$ on $\mathbb{R} \times \mathbb{C}$ by

Definition 18. For $\psi \in C^\infty_c(\mathbb{R} \times \mathbb{C})$, set

$$\langle H^z_{\tau p}, \psi \rangle = \lim_{\epsilon \to 0} \int_\epsilon^\infty \int_{\mathbb{C}} H_{\tau p}(s, z, w)\psi(s, w)\, dw\, ds.$$

The kernel of this distribution is

$$H^z_{\tau p}(s, w) = \begin{cases} H_{\tau p}(s, z, w) & \text{if } s > 0, \\ 0 & \text{if } s \leqslant 0. \end{cases}$$

We need to prove that the limit defining $H^z_{\tau p}$ exists and defines a distribution on $\mathbb{R} \times \mathbb{C}$. To do so, we need the following lemma which gives control over pointwise bounds on $|e^{-s\,\Box_{\tau p}} f - f|$.

Lemma 19. *There is a constant depending only on the degree of p so that if $f \in \mathrm{Dom}(\Box^j_{\tau p})$ for $j \leqslant 2$, then for any $z \in \mathbb{C}$ and $0 < R < R_{\tau p}(z)$*

$$\sup_{D(z,R)} \left| f(w) - e^{-s\,\Box_{\tau p}}[f](w) \right| \leqslant C \frac{s}{R} \left(\left\| \Box_{\tau p}[f] \right\|_{L^2(\mathbb{C})} + R^2 \left\| \Box^2_{\tau p}[f] \right\|_{L^2(\mathbb{C})} \right).$$

Proof. Let $f \in \mathrm{Dom}(\Box^j_{\tau p})$, $0 \leqslant j \leqslant 2$. By Theorems 13 and 5(c), we have

$$\begin{aligned} \sup_{D(z,R)} \left| f(w) - e^{-s\,\Box_{\tau p}}[f](w) \right| &\leqslant C \frac{1}{R} \sum_{j=0}^{1} R^{2j} \left\| \Box^j_{\tau p}[f] - \Box^j_{\tau p}\left[e^{-s\,\Box_{\tau p}} f\right] \right\|_{L^2(\mathbb{C})} \\ &= \frac{C}{R} \sum_{j=0}^{1} R^{2j} \left\| \left(I - e^{-s\,\Box_{\tau p}}\right)\left[\Box^j_{\tau p} f\right] \right\|_{L^2(\mathbb{C})} \\ &\leqslant C \frac{s}{R} \sum_{j=0}^{1} R^{2j} \left\| \Box^{j+1}_{\tau p}[f] \right\|_{L^2(\mathbb{C})}. \qquad \Box \end{aligned}$$

Lemma 20. *For each $z \in \mathbb{C}$, the limit defining $H^z_{\tau p}$ exists and defines a distribution on $\mathbb{R} \times \mathbb{C}$.*

Proof. Let $\psi \in C^\infty_c(\mathbb{R} \times \mathbb{C})$. Then there is a closed, bounded interval $I \subset \mathbb{R}$ and a compact set $K \subset \mathbb{C}$ so that $\operatorname{supp} \psi \in I \times K$. Set $\psi_s(z) = \psi(s, z)$. Then $\{\psi_s\} \subset C^\infty_c(\mathbb{R})$ with each element having support in K. If $0 < \epsilon_1 < \epsilon_2$, then

$$\begin{aligned} \int_{\epsilon_1}^{\epsilon_2} H_{\tau p}(s, z, w)\psi(s, w)\, dw\, ds &= \int_{\epsilon_1}^{\epsilon_2} e^{-s\,\Box_{\tau p}}[\psi_s](z)\, ds \\ &= \int_{\epsilon_1}^{\epsilon_2} e^{-s\,\Box_{\tau p}}[\psi_s](z) - \psi(s, z)\, ds + \int_{\epsilon_1}^{\epsilon_2} \psi(s, z)\, ds. \end{aligned}$$

From Lemma 19 and Hölder's inequality, we have (with $R < R_{\tau p}(z)$),

$$\left|\int_{\epsilon_1}^{\epsilon_2}\int_{\mathbb{C}} H_{\tau p}(s,z,w)\psi(s,w)\,dw\,ds\right| \leqslant C\frac{\epsilon_2}{R}\sum_{j=0}^{1} R^{2j}\int_0^{\epsilon_2}\left\|\Box_{\tau p}^{j+1}[\psi_s]\right\|_{L^2(\mathbb{C})}\,ds + \epsilon_2\|\psi\|_{L^\infty(\mathbb{R}\times\mathbb{C})}$$
$$\leqslant C\frac{\epsilon_2^{3/2}}{R}\sum_{j=0}^{1}\left\|\Box_{\tau p}^{j+1}[\psi]\right\|_{L^2(\mathbb{R}\times\mathbb{C})} + \epsilon_2\|\psi\|_{L^\infty(\mathbb{R}\times\mathbb{C})}.$$

These last terms go to 0 as $\epsilon_2 \to 0$, so the limit defining $H_{\tau p}^z$ there exists. □

Theorem 21. *In $\mathcal{D}'(\mathbb{R}\times\mathbb{C})$,*

$$\left(\partial_s + \Box_{\tau p,w}^{\#}\right)\left(H_{\tau p}^z\right) = \delta_0 \otimes \delta_z.$$

Proof. Let $\psi \in C_c^\infty(\mathbb{R}\times\mathbb{C})$. Then

$$\left\langle\left(\partial_s + \Box_{\tau p,w}^{\#}\right)\left(H_{\tau p}^z\right), \psi\right\rangle = \left\langle H_{\tau p}^z, (-\partial_s + \Box_{\tau p,w})\psi\right\rangle$$
$$= -\lim_{\epsilon\to 0}\int_\epsilon^\infty\int_{\mathbb{C}} H_{\tau p}(s,z,w)\partial_s\psi(s,w)\,dw\,ds$$
$$+ \lim_{\epsilon\to 0}\int_\epsilon^\infty\int_{\mathbb{C}} H_{\tau p}(s,z,w)\,\Box_{\tau p,w}\,\psi(w,s)\,dw\,ds.$$

Since s is bounded away from 0 and $H_{\tau p} \in C^\infty((0,\infty)\times\mathbb{C}\times\mathbb{C})$, the first term yields

$$-\int_\epsilon^\infty\int_{\mathbb{C}} H_{\tau p}(s,z,w)\frac{\partial\psi}{\partial s}(s,w)\,dw\,ds$$
$$= -\int_\epsilon^\infty\frac{\partial}{\partial s}\int_{\mathbb{C}} H_{\tau p}(s,z,w)\psi(s,w)\,dw\,ds + \int_\epsilon^\infty\int_{\mathbb{C}}\frac{\partial}{\partial s}H_{\tau p}(s,z,w)\psi(s,w)\,dw\,ds$$
$$= \int_{\mathbb{C}} H_{\tau p}(\epsilon,z,w)\psi(\epsilon,w)\,dw + \int_\epsilon^\infty\int_{\mathbb{C}}\frac{\partial}{\partial s}H_{\tau p}(s,z,w)\psi(s,w)\,dw\,ds.$$

Also,

$$\int_\epsilon^\infty\int_{\mathbb{C}} H_{\tau p}(s,z,w)\,\Box_{\tau p,w}\,\psi(s,w)\,dw\,ds = \int_\epsilon^\infty\int_{\mathbb{C}}\Box_{\tau p,w}^{\#}\,H_{\tau p}(s,z,w)\psi(s,w)\,dw\,ds.$$

Using Theorem 6(b) and adding our equalities together, we have

$$\int_{\epsilon}^{\infty}\int_{\mathbb{C}} -H_{\tau p}(s,z,w)\partial_s\psi(s,w) + H_{\tau p}(s,z,w)\,\Box_{\tau p,w}\,\psi(s,w)\,dw\,ds$$

$$= \int_{\mathbb{C}} H_{\tau p}(\epsilon,z,w)\psi(\epsilon,w)\,dw + \int_{\epsilon}^{\infty}\int_{\mathbb{C}} \big(\partial_s + \Box^{\#}_{\tau p,w}\big)H_{\tau p}(s,z,w)\psi(s,w)\,dw\,ds$$

$$= \int_{\mathbb{C}} H_{\tau p}(\epsilon,z,w)\psi(\epsilon,w)\,dw.$$

Hence

$$\big\langle\big(\partial_s + \Box^{\#}_{\tau p,w}\big)\big[H^z_{\tau p}\big],\psi\big\rangle = \lim_{\epsilon\to 0}\int_{\mathbb{C}} H_{\tau p}(\epsilon,z,w)\psi(\epsilon,w)\,dw$$

$$= \lim_{\epsilon\to 0} e^{-\epsilon\,\Box_{\tau p}}[\psi_\epsilon](z) = \psi(0,z) = \langle\delta_0\otimes\delta_z,\psi\rangle. \qquad \Box$$

9. Estimates on $\frac{\partial^n}{\partial s^n} Y^\alpha H_{\tau p}(s,z,w)$

In this section, we prove Theorem 1, the result on pointwise estimates of $|X^I_z U^J_w H_{\tau p}(s,z,w)|$. We begin the section with a study of how the heat kernel behaves under scaling.

9.1. Scaling and the heat kernel

The structure of $\Box_{\tau p}$ is critical in this section. Expanding $\Box_{\tau p}$, we have

$$\Box_{\tau p} = -\left(\frac{\partial}{\partial\bar z} + \tau\frac{\partial p}{\partial\bar z}\right)\left(\frac{\partial}{\partial z} - \tau\frac{\partial p}{\partial z}\right)$$

$$= -\frac{\partial^2}{\partial z\partial\bar z} + \tau\frac{\partial^2 p}{\partial z\partial\bar z} + \tau^2\frac{\partial p}{\partial z}\frac{\partial p}{\partial\bar z} + \tau\left(\frac{\partial p}{\partial z}\frac{\partial}{\partial\bar z} - \frac{\partial p}{\partial\bar z}\frac{\partial}{\partial z}\right) \tag{16}$$

$$= -\frac{1}{4}\Delta + \frac{1}{4}\tau\Delta p + \frac{\tau^2}{4}|\nabla p|^2 + \frac{i}{2}\tau\left(\frac{\partial p}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial p}{\partial x_2}\frac{\partial}{\partial x_1}\right). \tag{17}$$

Let $p_0(w) = p(w)$ and fix $z_0\in\mathbb{C}$. Let $p_1(w) = p_0(w+z_0)$. Our first scaling result is as follows.

Proposition 22.

$$H_{\tau p_0}(s, z+z_0, w+z_0) = H_{\tau p_1}(s,z,w).$$

Proof. Fix $z_0\in\mathbb{C}$. Let $A_{z_0}[f](z) = f(z-z_0)$. A_{z_0} is an isometry on $L^2(\mathbb{C})$, and

$$\mathcal{H}_{\tau p_1}[f](z) = -\frac{1}{4}\Delta f(z) + \frac{\tau}{4}\Delta p_0(z+z_0)f(z) + \frac{\tau^2}{4}\big|\nabla p_0(z+z_0)\big|^2 f(z)$$

$$+\frac{i}{2}\tau\left(\frac{\partial p_0}{\partial x_1}(z+z_0)\frac{\partial f}{\partial x_2}(z)-\frac{\partial p_0}{\partial x_2}(z+z_0)\frac{\partial f}{\partial x_1}(z)\right)$$

$$=A_{z_0}^{-1}\biggl[-\frac{1}{4}\Delta f(z-z_0)+\frac{\tau}{4}\Delta p_0(z)f(z-z_0)+\frac{\tau^2}{4}\bigl|\nabla p_0(z)\bigr|^2 f(z-z_0)$$

$$+\frac{i}{2}\tau\left(\frac{\partial p_0}{\partial x_1}(z)\frac{\partial f}{\partial x_2}(z-z_0)-\frac{\partial p_0}{\partial x_2}(z)\frac{\partial f}{\partial x_1}(z-z_0)\right)\biggr]$$

$$=A_{z_0}^{-1}\mathcal{H}_{\tau p_0}A_{z_0}[f](z).$$

Also, if $\psi\in C_c^\infty(\mathbb{C}\times\mathbb{R})$,

$$A_{z_0}^{-1}(\delta_0\otimes\delta_z)A_{z_0}\psi(s,w)=A_{z_0}^{-1}\psi(0,z-z_0)=\psi(0,z),$$

and

$$\begin{aligned}A_{z_0}^{-1}(\delta_0\otimes\delta_z)A_{z_0}\psi(s,w)&=A_{z_0}^{-1}\mathcal{H}_{\tau p_0}\int_0^\infty\int_{\mathbb{C}}H_{\tau p_0}(s,z,w)\psi(s,w-z_0)\,dw\,ds\\&=A_{z_0}^{-1}\int_0^\infty\int_{\mathbb{C}}\left(\frac{\partial}{\partial s}+\Box_{\tau p_0,z}\right)H_{\tau p_0}(s,z,w)\psi(s,w-z_0)\,dw\,ds\\&=A_{z_0}^{-1}\int_0^\infty\int_{\mathbb{C}}\left(\frac{\partial}{\partial s}+\Box^\#_{\tau p_0,w}\right)H_{\tau p_0}(s,z,w)\psi(s,w-z_0)\,dw\,ds\\&=A_{z_0}^{-1}\int_0^\infty\int_{\mathbb{C}}\left(\frac{\partial}{\partial s}+\Box^\#_{\tau p_1,w}\right)H_{\tau p_0}(s,z,w+z_0)\psi(s,w-z_0)\,dw\,ds\\&=\int_0^\infty\int_{\mathbb{C}}\left(\frac{\partial}{\partial s}+\Box^\#_{\tau p_1,w}\right)H_{\tau p_0}(s,z+z_0,w+z_0)\psi(s,w-z_0)\,dw\,ds.\end{aligned}$$

Thus, $H_{\tau p_0}(s,z+z_0,w+z_0)=H_{\tau p_1}(s,z,w)$. □

The method of proof in Proposition 22 also proves the following two scaling results.

Proposition 23. *If* $z_0\in\mathbb{C}$ *and*

$$p_2^{z_0}(w)=\sum_{j,k\geqslant 1}a^z_{jk}(w-z_0)^j\,\overline{(w-z_0)}^k,$$

then

$$H_{\tau p_2^{z_0}}(s,z,w)=e^{i\tau T(w,z_0)}H_{\tau p_1}(s,z_0,w).$$

Let $T_\lambda \psi(s,w) = \lambda^2 \psi(\lambda^2 s, \lambda w)$ and $T_\lambda f(w) = \lambda f(\lambda w)$ act on functions on $\mathbb{R} \times \mathbb{C}$ and $\mathbb{C}$, respectively. In either case, T_λ is an isometry on L^2. Our final proposition in this section investigates conjugating $\mathcal{H}_{\tau p}$ by T_λ. Let $\psi_\lambda(s,w) = \psi(\lambda^2 s, \lambda w)$ and $f_\lambda(w) = f(\lambda w)$.

Proposition 24. *If $p_3^\lambda = p_2(z/\lambda)$, then*

$$\frac{1}{\lambda^2} H_{\tau p_2^{z_0}}\big(s/\lambda^2, z/\lambda, w/\lambda\big) = H_{\tau p_3^\lambda}(s,z,w).$$

9.2. Pointwise estimates of $|H_{\tau p}(s,z,w)|$

We first show that $|H_{\tau p}(s,z,w)|$ has Gaussian decay. To do so, we will find it convenient to work in real variable notation instead of complex notation. As such, let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Our first goal is to prove the following theorem.

Theorem 25. *If $e^{-s\,\square_{\tau p}}[f](x) = \int_{\mathbb{C}} H_{\tau p}(s,x,y) f(y)\,dy$, then the heat kernel, $H_{\tau p}(s,x,y)$, satisfies the estimate*

$$\big|H_{\tau p}(s,x,y)\big| \leqslant \frac{1}{\pi s} e^{-\frac{|x-y|^2}{s}}.$$

Proof. We will use the Feynman–Kac–Itô formula from [24]. Let dx be Lebesgue measure on $\mathbb{R}^2$ and let $(B, \mathfrak{B}, dP)$ be a measure space of sample paths for a 2-dimensional Brownian motion $b(s)$. Let $d\mu = dP \otimes dx$ be Wiener measure on $B \times \mathbb{R}^2$ and let $\omega(s) = x + b(s)$. If we let

$$a(x) = \tau\left(-\frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_1}\right) \quad \text{and} \quad V(x) = \frac{\tau}{2}\Delta p(x),$$

then for $f \in C^2(\mathbb{R}^2)$,

$$\frac{1}{2}(-i\nabla - a)^2 f + Vf = -\frac{1}{2}\Delta f + \frac{i}{2}(\nabla \cdot a) f + i a \cdot \nabla f + \frac{1}{2}|a|^2 f + Vf.$$

But

$$\nabla \cdot a = \tau\left(-\frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{\partial^2 p}{\partial x_1 \partial x_2}\right) = 0 \quad \text{and} \quad \frac{1}{2}|a|^2 = \frac{1}{2}\tau^2 |\nabla p|^2.$$

Thus,

$$\frac{1}{2}(-i\nabla - a)^2 f + Vf = 2\square_{\tau p},$$

so $2\,\square_{\tau p}$ is the quantum mechanical energy operator for a particle in a magnetic field with vector potential $a(x)$ and electric potential V. The Feynman–Kac–Itô formula for $f, g \in C_c^\infty(\mathbb{R}^2)$ is

$$\big(e^{-2s\,\square_{\tau p}} f, g\big) = \int e^{F(s,\omega)} f\big(\omega(s)\big)\overline{g\big(\omega(0)\big)}\,d\mu, \tag{18}$$

where

$$F(s,\omega) = -i\int_0^s a(\omega(t))\cdot d\omega(t) - \frac{i}{2}\int_0^s (\nabla\cdot a)(\omega(t))\,dt - \int_0^s V(\omega(t))\,dt.$$

$b(s)$ has 2-dimensional normal distribution with covariance s, so we can rewrite (18) as follows:

$$\begin{aligned}
&\iint_{\mathbb{R}^2\times\mathbb{R}^2} H_{\tau p}(2s,x,y) f(y)\overline{g(x)}\,dy\,dx \\
&\quad = \int e^{F(s,\omega)} f(\omega(s))\overline{g(\omega(0))}\,d\mu \\
&\quad = \int_{\mathbb{R}^2} E\big[e^{F(s,\omega)} f(\omega(s))\overline{g(\omega(0))}\big]\,dx \\
&\quad = \int_{\mathbb{R}^2} E\big[E\big[e^{F(s,\omega)} f(\omega(s))\overline{g(\omega(0))}\,\big|\,\omega(0)=x,\omega(s)=x+y\big]\big]\,dx \\
&\quad = \int_{\mathbb{R}^2} E\big[E\big[e^{F(s,\omega)} f(\omega(s))\,\big|\,\omega(0)=x,\omega(s)=x+y\big]\big]\overline{g(x)}\,dx \\
&\quad = \frac{1}{2\pi s}\iint_{\mathbb{R}^2\times\mathbb{R}^2} e^{\tilde F(s,x+y)} f(x+y)\overline{g(x)}e^{-\frac{|y|^2}{2s}}\,dy\,dx \\
&\quad = \frac{1}{2\pi s}\iint_{\mathbb{R}^2\times\mathbb{R}^2} e^{\tilde F(s,y)} f(y)\overline{g(x)}e^{-\frac{|x-y|^2}{2s}}\,dy\,dx.
\end{aligned}$$

Thus,

$$H_{\tau p}(2s,x,y) = \frac{1}{2\pi s}e^{\tilde F(s,y)}e^{-\frac{|x-y|^2}{2s}}$$

for some $\tilde F(s,y)$ satisfying $|e^{\tilde F(s,y)}| \leqslant 1$. □

A critically important fact about the Feynman–Kac–Itô formula is the requirement that $V \geqslant 0$. When $\tau < 0$, $V \leqslant 0$, and the argument from Theorem 25 fails. Even if we could use the argument, the real part of $e^{F(s,\omega)}$ is

$$e^{-\int_0^s V(\omega(t))\,dt},$$

a term that we would expect to be very large. In fact, when $\tau < 0$ $H_{\tau p}(s,z,w)$ only satisfies Gaussian decay near $s = 0$ [20], so an analog to Theorem 25 is false.

We now turn to proving a large time decay estimate for $H_{\tau p}(s,z,w)$. Let $\mathfrak{P}$ be the set of polynomials of $\deg(p)$ whose coefficients (in absolute value) sum to 1. We can identify the set of polynomials of $\deg(p)$ with $\mathbb{R}^n$ for some n, and under this identification, $\mathfrak{P}$ is identified with the

ℓ^1 unit sphere, a compact set. Having constants depending only on $\mathfrak{P}$ is essential for estimates obtained through scaling.

Theorem 26. *If $e^{-s\,\Box_{\tau p}}[f](z) = \int_{\mathbb{C}} H_{\tau p}(s,z,w) f(w)\,dw$, then there exist constants C_1 and C_2 which depend on the degree of p so that*

$$\left|H_{\tau p}(s,z,w)\right| \leqslant \frac{C_1}{s} e^{-C_2 \frac{s}{\mu(z,1/\tau)^2}} e^{-C_2 \frac{s}{\mu(w,1/\tau)^2}}.$$

Proof. By the conjugate symmetry of $H_{\tau p}$, it is enough to show the bound $\frac{C_1}{s} e^{-C_2 \frac{s}{\mu(z,1/\tau)^2}}$. From Proposition 22, there exists a polynomial $p_1(w) = p_0(w+z)$ so that

$$H_{\tau p}(s,z,w) = H_{\tau p_1}(s,0,w-z),$$

so we can reduce to the case of estimating $H_{\tau p_1}(s,0,w)$. By Proposition 23, for

$$p_2(w) = \sum_{j,k\geqslant 1} a^0_{jk} w^j \bar{w}^k,$$

we have

$$e^{i\tau T(w,0)} H_{\tau p_1}(s,0,w) = H_{\tau p_2}(s,0,w),$$

so it is enough to estimate $|H_{\tau p_2}(s,0,w)|$. Observe that $p_2(w)$ has the property

$$\frac{\partial^k p_2}{\partial z^k}(0) = \frac{\partial^k p_2}{\partial \bar{z}^k}(0) = 0 \quad \text{for all } k.$$

If we set $\lambda = \mu(z,1/\tau)^{-1}$ and $p_3(w) = p_2(w/\lambda)$, then $p_3 \in \mathfrak{P}$ since

$$\tau \sum_{j,k\geqslant 1} \frac{1}{j!k!} \left|\frac{\partial^{j+k} p_2}{\partial z^j \partial \bar{z}^k}(0)\right| z^j \bar{z}^k \lambda^{-j-k} = \tau \Lambda\left(0, \mu\left(0, \frac{1}{\tau}\right)\right) \sim 1.$$

From Proposition 24,

$$\frac{1}{\lambda^2} H_{\tau p_2}\left(\frac{s}{\lambda^2}, 0, \frac{w}{\lambda}\right) = H_{p_3}(s,0,w).$$

We now estimate $|H_{p_3}(s,w,0)|$. Let $h(s,w) = H_{p_3}(s,w,0)$. By Theorem 6(b), $\partial h/\partial s = \overline{Z}_{p_3} Z_{p_3} h$. Let $g(s) = \int_{\mathbb{C}} |h(s,w)|^2\,dw$. From [4], there exists $C = C_{\mathfrak{P}}$ so that $\|f\|_{L^2(\mathbb{C})} \leqslant C\|Z_{p_3} f\|_{L^2(\mathbb{C})}$. Thus

$$\begin{aligned} g'(s) &= \int_{\mathbb{C}} \frac{d}{ds}\bigl(h(s,w)\overline{h(s,w)}\bigr)\,dw = 2\,\mathrm{Re} \int_{\mathbb{C}} \frac{\partial h}{\partial s}(s,w)\overline{h(s,w)}\,dw \\ &= 2\,\mathrm{Re} \int_{\mathbb{C}} \overline{Z}_{p_3} Z_{p_3} h(s,w)\overline{h(s,w)}\,dw = -2 \int_{\mathbb{C}} \left|Z_{p_3} h(s,w)\right|^2 dw \end{aligned}$$

$$\leqslant -C \int_{\mathbb{C}} \left|h(s,w)\right|^2 dw = -Cg(s).$$

Since $g(s) > 0$, $g'(s)/g(s) \leqslant -C$, and integrating from $s/2$ to s, we have

$$g(s) \leqslant g\left(\frac{s}{2}\right)e^{-Cs} \leqslant C_1 \frac{e^{-Cs}}{s},$$

where the last inequality follows from Theorem 25. The constant C_1 does not depend on p_3 (or $\mathfrak{P}$).

Next, $e^{-s\Box_{p_3}}$ is a semigroup, so $e^{-s\Box_{p_3}} e^{-s\Box_{p_3}} f(z) = e^{-2s\Box_{p_3}} f(z)$. On the kernel side, this means we have the reproducing identity

$$H_{p_3}(2s,z,w) = \int_{\mathbb{C}} H_{p_3}(s,z,v) H_{p_3}(s,v,w)\,dv,$$

and an application of Cauchy–Schwarz yields

$$\left|H_{p_3}(2s,0,w)\right| \leqslant \left(\int_{\mathbb{C}} \left|H_{p_3}(s,0,v)\right|^2 dv\right)^{1/2} \left(\int_{\mathbb{C}} \left|H_{p_3}(s,v,w)\right|^2 dv\right)^{1/2} \leqslant C_1 \frac{e^{-Cs}}{s}.$$

Undoing the scaling finishes the proof. $\Box$

The motivation for using $g'(s)$ and the reproducing identity was [6].

9.3. Derivative estimates

The derivative estimates are proven in a series of lemmas. The most accessible case is proven first and each successive lemma builds on the previous calculation. Each L^2 estimate at one step is used to prove a pointwise estimate in the next. Define the decay term $D(s,x,y)$ to be

$$D(s,x,y) = e^{-\frac{|x-y|^2}{2s}} e^{-C_2 \frac{s}{\mu(x,1/\tau)^2}} e^{-C_2 \frac{s}{\mu(y,1/\tau)^2}}, \tag{19}$$

where C_2 is the constant from Theorem 26. Also, let

$$I_r(s) = \left(s - r^2, s\right) \quad \text{and} \quad Q_r(s,x) = I_r(s) \times D(x,r).$$

We need a version of the subsolution estimate from [11].

Lemma 27. *Let* $(s_0, z_0) \in (0,\infty) \times \mathbb{C}$ *and* $u(s,z)$ *be a* C^2 *solution of*

$$\frac{\partial u}{\partial s} + \Box_{\tau p} u = 0$$

on $Q_{2r}(s_0, z_0)$. *If* $\tau > 0$, *then there exists* $C > 0$ *so that*

$$\sup_{(s,z)\in Q_{r/2}(s_0,z_0)} \left|u(s,z)\right| \leqslant \frac{C}{r^2} \iint\limits_{Q_{2r/3}(s_0,z_0)} \left|u(s,z)\right|^2 \, dz\, ds.$$

Proposition 28. *There exists* C_n *so that for* $0 < r < \sqrt{s_0}/16$,

$$\left\| \frac{\partial^n H_{\tau p}}{\partial s^n}(\cdot, x, \cdot) \right\|_{L^2(Q_r(s_0,y_0))} \leqslant \frac{C}{s_0^n}.$$

Proof. We have

$$\begin{aligned}
\left\| \frac{\partial^n H_{\tau p}}{\partial s^n}(\cdot, x_0, \cdot) \right\|^2_{L^2(Q_r(s_0,y_0))} &= \int\limits_{I_r(s_0)} \left| \left(\int\limits_{D(y_0,r)} \left| \frac{\partial^n H_{\tau p}}{\partial s^n}(s, x_0, y) \right|^2 dy \right)^{1/2} \right|^2 ds \\
&= \int\limits_{I_r(s_0)} \left| \sup_{\substack{\varphi \in C_c^\infty(D(y_0,r)) \\ \|\varphi\|_{L^2}=1}} \int \frac{\partial^n H_{\tau p}}{\partial s^n}(s, x_0, y)\varphi(y)\, dy \right|^2 ds \\
&= \int\limits_{I_r(s_0)} \left| \sup_{\substack{\varphi \in C_c^\infty(D(y_0,r)) \\ \|\varphi\|_{L^2}=1}} \frac{\partial^n}{\partial s^n} H^s_{\tau p}[\varphi](x_0) \right|^2 ds.
\end{aligned} \tag{20}$$

The key to the proof is that $\frac{\partial^n}{\partial s^n} H^s_{\tau p}[\varphi](x)$ satisfies

$$\left(\frac{\partial}{\partial s} + \Box_{\tau p, x} \right) \frac{\partial^n}{\partial s^n} H^s_{\tau p}[\varphi](x) = 0.$$

By Lemma 27 and Theorem 5(d), estimating an arbitrary term from the supremum in (20) yields

$$\begin{aligned}
\left| \frac{\partial^n}{\partial s^n} H^s_{\tau p}[\varphi](x_0) \right| &\leqslant \frac{C}{r^2} \left(\iint\limits_{Q_r(s,x_0)} \left| \frac{\partial^n}{\partial t^n} H^t_{\tau p}[\varphi](x) \right|^2 dx\, dt \right)^{1/2} \\
&\leqslant \frac{C}{r^2} \left(\int\limits_{I_{\sqrt{2}r}(s_0)} \left\| \frac{\partial^n}{\partial t^n} H^t_{\tau p}[\varphi] \right\|^2_{L^2(D(x_0,r))} dt \right)^{1/2} \\
&\leqslant \frac{C}{r^2} \left(\int\limits_{s_0-2r^2}^{s_0} \frac{1}{t^{2n}}\, dt \right)^{1/2} \leqslant \frac{C}{r s_0^n}.
\end{aligned} \tag{21}$$

Putting (20) into (21), we have

$$\left\| \frac{\partial^n H_{\tau p}}{\partial s^n}(\cdot, x_0, \cdot) \right\|_{L^2(Q_r(s_0,y_0))} \leqslant C \left(\int\limits_{I_r(s_0)} \frac{1}{r^2 s_0^{2n}}\, ds \right)^{1/2} = \frac{C}{s_0^n}. \qquad \Box$$

Lemma 29. *Let n_1, n_2, $n_3 \geqslant 0$ and $n = n_1 + n_2 + n_3$. Then there exists $C_n > 0$ so that*

$$\left|\frac{\partial^{n_1}}{\partial s^{n_1}} \Box^{n_2}_{\tau p,x}\left(\Box^{\#}_{\tau p,y}\right)^{n_3} H_{\tau p}(s,x,y)\right| \leqslant \frac{C_n}{s_0^{n+1}} D(s,x,y)^{1/2}.$$

Proof. By Theorem 6, it is enough to show the estimate for $H_n(s,x,y) = \frac{\partial^n}{\partial s^n} H_{\tau p}(s,x,y)$. Proof by induction. The base case follows from combining Theorems 25 and 26:

$$\left|H_{\tau p}(s,x,y)\right| \leqslant \left|H_{\tau p}(s,x,y)\right|^{1/2}\left|H_{\tau p}(s,x,y)\right|^{1/2} \leqslant \frac{C}{s} D(s,x,y).$$

Assume the result holds for H_{n-1}. Let $r = \sqrt{s_0}/16$. Let $\psi \in C_c^\infty(Q_{2r}(s_0,y_0))$ where $\psi|_{Q_r(s_0,y_0)} \equiv 1$, $0 \leqslant \psi \leqslant 1$, and $\partial^j\psi/\partial s^j \leqslant c_j/r^{2j}$. We can use Lemma 27 because if $s > 0$, $H_{n-1}(s,z,w)$ satisfies $\mathcal{H}_{\tau p} H_{n-1}(s,x,y) = 0$. Using Lemma 27 and Proposition 28, for $r > 0$ and $Q = Q_{2r}(s_0,y_0)$

$$\begin{aligned}
\left|\frac{\partial^n H_{\tau p}}{\partial s^n}(s_0,x,y_0)\right| &\leqslant \frac{C}{r^2}\left(\iint\limits_{Q_r(s_0,y_0)} \left|\frac{\partial^n H_{\tau p}}{\partial s^n}(s,x,y)\right|^2 ds\,dy\right)^{1/2} \\
&\leqslant \frac{C}{r^2}\left(\iint\limits_{\mathbb{R}\times\mathbb{C}} \frac{\partial^n H_{\tau p}}{\partial s^n}(s,x,y)\overline{\frac{\partial^n H_{\tau p}}{\partial s^n}(s,x,y)}\psi(s,y)\,ds\,dy\right)^{1/2} \\
&= \frac{C}{r^2}\left(\iint\limits_{\mathbb{R}\times\mathbb{C}} \overline{H_{\tau p}(s,x,y)}\sum_{j=0}^{n} \frac{\partial^{n+j} H_{\tau p}}{\partial s^{n+j}}(s,x,y)\frac{\partial^{n-j}\psi}{\partial s^{n-j}}(s,y)\,ds\,dy\right)^{1/2} \\
&\leqslant \frac{C}{r^2}\left[\left\|H_{\tau p}(\cdot,x,\cdot)\right\|_{L^2(Q)}\sum_{j=0}^{n} c_j\frac{1}{r^{2(n-j)}}\left\|\frac{\partial^{n+j} H_{\tau p}}{\partial s^{n+j}}(\cdot,x,\cdot)\right\|_{L^2(Q)}\right]^{1/2} \\
&\leqslant \frac{C}{r^2}\left[H_{\tau p}(s_0,x,y_0)r^2\left(\frac{1}{s_0^{2n}} + \frac{1}{r^{2n}s_0^n}\right)\right]^{1/2} \\
&\leqslant \frac{C_n}{r}\frac{D(s_0,x,y_0)^{1/2}}{s_0^{1/2}}\left(\frac{1}{s_0^n} + \frac{1}{r^n s_0^{n/2}}\right) \leqslant \frac{C_n}{s_0^{n+1}} D(s_0,x,y_0)^{1/2}. \qquad \Box
\end{aligned}$$

Integrating in y gives the immediate corollary.

Corollary 30. *Let n_1, n_2, $n_3 \geqslant 0$ and $n = n_1 + n_2 + n_3$. Then there exists $C_n > 0$ so that*

$$\left\|\frac{\partial^{n_1}}{\partial s^{n_1}} \Box^{n_2}_{\tau p,x}\left(\Box^{\#}_{\tau p,y}\right)^{n_3} H_{\tau p}(s,x,\cdot)\right\|_{L^2(\mathbb{C})} \leqslant \frac{C_n}{s^{n+1/2}}.$$

Lemma 31. *Let α be a multi-index and $j \geqslant 0$. Then there exists $C_{|\alpha|,j} > 0$ so that if $R = \min\{\frac{\sqrt{s_0}}{16}, \frac{\mu(x_0,1/\tau)}{4}\}$, then*

$$\left|X_x^\alpha\left(\Box^{\#}_{\tau p,y}\right)^j H_{\tau p}(s,x,y)\right| + \left|\Box^j_{\tau p,x} U_y^\alpha H_{\tau p}(s,x,y)\right| \leqslant \frac{C_{|\alpha|}}{R^{\frac12+\frac12|\alpha|}s^{\frac34+j+\frac14|\alpha|}} D(s,x_0,y)^{\frac14}.$$

Proof. It is enough to bound $|U_y^\alpha \Box_{\tau p,x}^j H_{\tau p}(s, x_0, y)|$ for a fixed $x_0 \in \mathbb{C}$. In fact, we can even assume that

$$\frac{\partial^n p}{\partial z^n}(x_0) = \frac{\partial^n p}{\partial \bar{z}^n}(x_0) = 0 \quad \text{for all } n$$

by Proposition 23. This means if $|y - x_0| \leqslant \mu(x_0, 1/\tau)$,

$$\left|\frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(y)\right| \leqslant \frac{1}{\mu(x_0, 1/\tau)^{j+k}} \tau \Lambda\big(x_0, \mu(x_0, 1/\tau)\big) \sim \frac{1}{\mu(x_0, 1/\tau)^{j+k}}. \tag{22}$$

Let $R = \min\{\frac{\sqrt{s}}{16}, \frac{1}{4}\mu(x_0, \frac{1}{\tau})\}$. Also, fix s and let $g(y) = \Box_{\tau p,x}^j H_{\tau p}(s, x_0, y)$. Let D stand for $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$. Then from Theorem 13, if $\varphi \in C_c^\infty(D(x_0, R))$ with $0 \leqslant \varphi \leqslant 1$, $|D^\beta \varphi| \leqslant \frac{c_{|\beta|}}{R^{|\beta|}}$ for $0 \leqslant |\beta| \leqslant 2$, we have

$$\big|U_y^\alpha g(y)\big| \leqslant \frac{C}{R} \sum_{|\beta| \leqslant 2} R^{|\beta|} \big\|\varphi^{\frac{1}{2}} U_y^\beta U_y^\alpha g\big\|_{L^2(\mathbb{C})}. \tag{23}$$

Then

$$\begin{aligned}\big\|\varphi^{\frac{1}{2}} U_y^\beta U_y^\alpha g\big\|_{L^2(\mathbb{C})}^2 &\leqslant \big(U_y^\beta U_y^\alpha g, \varphi U_y^\beta U_y^\alpha g\big) = \big|\big(g, U_y^\beta U_y^\alpha \big(\varphi U_y^\beta U_y^\alpha g\big)\big)\big| \\ &\leqslant \sum_{|\gamma_1|+|\gamma_2| = |\alpha|+|\beta|} c_{\gamma_1,\gamma_2} \|g\|_{L^2(D(x_0,R))} \frac{1}{R^{|\gamma_1|}} \big\|U_y^{\gamma_2} U_y^\beta U_y^\alpha g\big\|_{L^2(D(x_0,R))}.\end{aligned} \tag{24}$$

Next, from Corollary 30, Proposition 10, and Theorem 5, for some order 0 OPF operator B_τ, we have the estimate (note the complex conjugate in the first inequality),

$$\begin{aligned}\big\|U_y^{\gamma_2} U_y^\beta U_y^\alpha g\big\|_{L^2(D(x_0,R))}^2 &\leqslant \big(X_y^{\gamma_2} X_y^\beta X_y^\alpha \bar{g}, X_y^{\gamma_2} X_y^\beta X_y^\alpha \bar{g}\big) \\ &= \big|\big(\bar{g}, X_y^\alpha X_y^\beta X_y^{\gamma_2} X_y^{\gamma_2} X_y^\beta X_y^\alpha \bar{g}\big)\big| \\ &\leqslant \|g\|_{L^2(\mathbb{C})} \big\|B_\tau \Box_{\tau p}^{|\gamma_2|+|\beta|+|\alpha|} \bar{g}\big\|_{L^2(\mathbb{C})} \\ &\leqslant \frac{C}{s^{1/2+j}} s^{-(|\gamma_2|+|\beta|+|\alpha|+\frac{1}{2})}.\end{aligned} \tag{25}$$

Plugging (24) into (25) gives

$$\big\|\varphi^{\frac{1}{2}} X_y^\beta X_y^\alpha \bar{g}\big\|_{L^2(\mathbb{C})}^2 \leqslant C\big|g(y)\big|R \sum_{|\gamma_1|+|\gamma_2| = |\alpha|+|\beta|} c_{\gamma_1,\gamma_2} \frac{1}{R^{|\gamma_1|}} s^{-\frac{1}{2}(|\gamma_2|+|\beta|+|\alpha|+1+2j)}. \tag{26}$$

Using the fact that $R \leqslant \sqrt{s}$ and inserting (26) into (23), we have

$$\begin{aligned}
\left|X_y^\alpha \overline{g(y)}\right| &\leqslant \frac{C_{|\alpha|}}{R} \sum_{|\beta|\leqslant 2} R^{|\beta|}\left|g(y)\right|^{\frac{1}{2}} R^{\frac{1}{2}} \sum_{|\gamma_1|+|\gamma_2|=|\alpha|+|\beta|} s^{-\frac{1}{4}(|\gamma_2|+|\beta|+|\alpha|+1+2j)} \frac{1}{R^{|\gamma_1|/2}} \\
&\leqslant \frac{C_{|\alpha|}}{R^{1/2}s^{3/4+j}} D(s,x_0,y)^{\frac{1}{4}} \sum_{|\beta|\leqslant 2} \sum_{|\gamma_1|+|\gamma_2|=|\alpha|+|\beta|} R^{|\beta|-\frac{1}{2}|\gamma_1|} s^{-\frac{1}{4}|\gamma_2|} s^{-\frac{1}{4}|\beta|} s^{-\frac{1}{4}|\alpha|} \\
&\leqslant \frac{C_{|\alpha|}}{R^{1/2}s^{3/4+j}} D(s,x_0,y)^{\frac{1}{4}} R^{-\frac{1}{2}|\alpha|} s^{-\frac{1}{4}|\alpha|}. \qquad \square
\end{aligned}$$

Corollary 32. *Let α be a multi-index and $j \geqslant 0$. Then there exists $C_{|\alpha|,j} > 0$ so that*

$$\left\| X_x^\alpha \left(\Box_{\tau p,y}^{\#}\right)^j H_{\tau p}(s,x,\cdot)\right\|_{L^2(\mathbb{C})} + \left\| U_y^\alpha \Box_{\tau p,x}^j H_{\tau p}(s,x,\cdot)\right\|_{L^2(\mathbb{C})} \leqslant \frac{C_{|\alpha|,j}}{s^{\frac{1}{2}+j+\frac{|\alpha|}{2}}}.$$

Proof. Using the estimate from Lemma 31, if $R = \frac{\sqrt{s}}{16}$, then the result follows by direct calculation and a simple change of variables. If $R = \frac{1}{4}\mu(x, 1/\tau)$, then we use the fact that

$$D(s,x,y)^{\frac{1}{4}} \leqslant C_j D(s,x,y)^{\frac{1}{8}} \left(\frac{\mu(x,1/\tau)^2}{s}\right)^j \quad \text{for any } j \geqslant 0.$$

With this estimate, the result follows immediately. $\square$

The final lemma we need is as follows.

Lemma 33. *Let α and β be multi-indices. If $R = \min\{\frac{\sqrt{s_0}}{16}, \frac{\mu(x_0,1/\tau)}{4}\}$, then there exists $C_{|\alpha|,|\beta|} > 0$ so that*

$$\left|X_x^\alpha X_y^\beta H_{\tau p}(s,x,y)\right| \leqslant C_{|\alpha|,|\beta|} \frac{1}{R^{3/4}s^{5/8}} R^{-\frac{|\alpha|}{2}-\frac{|\beta|}{4}} s^{-\frac{|\alpha|}{4}-\frac{3|\beta|}{8}} D(s,x,y)^{\frac{1}{8}}.$$

Proof. As in Lemma 31, we may assume that

$$\frac{\partial^n p}{\partial z^n}(x_0) = \frac{\partial^n p}{\partial \bar{z}^n}(x_0) = 0 \quad \text{for all } n,$$

so by (22)

$$\left|\frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(y)\right| \lesssim \frac{1}{\mu(x_0, 1/\tau)^{j+k}}.$$

Fix s and x_0. Let $y \in D(x_0, R)$. Let $\varphi \in C_c^\infty(D(x_0, 2R))$ so that $\varphi|_{D(x_0,R)} \equiv 1$, $0 \leqslant \varphi \leqslant 1$, and $|D^\alpha \varphi| \leqslant c_{|\alpha|}/R^{|\alpha|}$. Let $f(x) = X_y^\beta H_{\tau p}(s,x,y)$ and $g(x) = X_x^\alpha X_y^\beta H_{\tau p}(s,x,y)$. From Theorem 13,

$$\left|g(x_0)\right| \leqslant \frac{C}{R} \sum_{|\gamma|\leqslant 2} R^{|\gamma|} \left\|\varphi^{\frac{1}{2}} X_x^\gamma g\right\|_{L^2(\mathbb{C})}.$$

Next,

$$\begin{aligned}\left\|\varphi^{\frac{1}{2}} X_x^\gamma g\right\|_{L^2(\mathbb{C})}^2 &= \left(X_x^\gamma X_x^\alpha f, \varphi X_x^\gamma g\right) = \left|\left(f, X_x^\alpha X_x^\gamma\left[\varphi X_x^\gamma g\right]\right)\right| \\ &= \sum_{|\gamma_1|+|\gamma_2|=|\gamma|+|\alpha|} c_{\gamma_1,\gamma_2}\left|\left(f, D^{\gamma_1}\varphi X_x^{\gamma_2} X_x^\gamma g\right)\right| \\ &\leqslant \sum_{|\gamma_1|+|\gamma_2|=|\gamma|+|\alpha|} c_{\gamma_1,\gamma_2}\|f\|_{L^2(D(x_0,R))}\frac{1}{R^{|\gamma_1|}}\left\|X_x^{\gamma_2}X_x^\gamma g\right\|_{L^2(\mathbb{C})}.\end{aligned}$$

Using Proposition 10 and Corollary 32, for some order zero OPF operator B_τ we have

$$\begin{aligned}\left\|X_x^{\gamma_2} X_x^\gamma g\right\|_{L^2(\mathbb{C})}^2 &= \left(X_x^{\gamma_2} X_x^\gamma X_x^\alpha f, X_x^{\gamma_2} X_x^\gamma X_x^\alpha f\right) \\ &= \left|\left(f, X_x^\alpha X_x^\gamma X_x^{\gamma_2} X_x^{\gamma_2} X_x^\gamma X_x^\alpha f\right)\right| \\ &\leqslant \|f\|_{L^2(\mathbb{C})}\|B_\tau \square_{\tau p}^{|\alpha|+|\gamma|+|\gamma_2|} f\|_{L^2(\mathbb{C})} \\ &= C_{|\alpha|+|\gamma|+|\gamma_2|+|\beta|} s^{-|\beta|-1-|\alpha|-|\gamma|-|\gamma_2|}.\end{aligned}$$

Thus, since $\|f\|_{L^2(D(x_0,R))} \leqslant C|f(x_0)|R$,

$$\begin{aligned}&\left|g(x_0)\right| \\ &\quad\leqslant C_{|\alpha|,|\beta|}\frac{1}{R}\sum_{|\gamma|\leqslant 2}\sum_{|\gamma_1|+|\gamma_2|=|\gamma|+|\alpha|} R^{|\gamma|}R^{-\frac{|\gamma_1|}{2}}s^{-\frac{1}{4}(|\beta|+1+|\alpha|+|\gamma|+|\gamma_2|)}\left|f(x_0)\right|^{\frac{1}{2}}R^{\frac{1}{2}} \\ &\quad\leqslant C_{|\alpha|,|\beta|}\frac{1}{R^{3/4}s^{5/8}}\sum_{|\gamma|\leqslant 2}\sum_{|\gamma_1|+|\gamma_2|=|\gamma|+|\alpha|} R^{|\gamma|}R^{-\frac{|\gamma_1|}{2}}s^{-\frac{|\gamma_2|}{4}}s^{-\frac{1}{4}(|\gamma|+|\alpha|)}s^{-\frac{|\beta|}{4}}s^{-\frac{|\beta|}{8}}R^{-\frac{|\beta|}{4}}D(s,x,y)^{\frac{1}{8}} \\ &\quad\leqslant C_{|\alpha|,|\beta|}\frac{1}{R^{3/4}s^{5/8}}R^{-\frac{|\alpha|}{2}}s^{-\frac{|\alpha|}{4}}R^{-\frac{|\beta|}{4}}s^{-\frac{3}{8}|\beta|}D(s,x,y)^{\frac{1}{8}}. \qquad \square\end{aligned}$$

As an immediate consequence of Lemma 33, we have:

Theorem 1. *Let p be a subharmonic, nonharmonic polynomial and $\tau > 0$ a parameter. If $n \geqslant 0$ and Y^α is a product of $|\alpha|$ operators $Y = \bar{Z}_{\tau p}$ or $Z_{\tau p}$ when acting in z and $\overline{(Z_{\tau p})}$ or $\overline{(\bar{Z}_{\tau p})}$ when acting in w, there exist constants $c, c_1 > 0$ independent of τ so that*

$$\left|\frac{\partial^n}{\partial s^n}Y^\alpha H_{\tau p}(s,z,w)\right| \leqslant c_1\frac{1}{s^{n+\frac{1}{2}|\alpha|+1}}e^{-\frac{|z-w|^2}{32s}}e^{-c\frac{s}{\mu(z,1/\tau)^2}}e^{-c\frac{s}{\mu(w,1/\tau)^2}}.$$

Also, c can be taken with no dependence on n and α.

Proof. The theorem follows Lemma 33 using the argument of the proof of Corollary 32. Also, by the argument of Lemma 33, specifically the argument of (23)–(25), we may take c to be independent of α and n. $\square$

Using Theorem 1, we can integrate in s and recover estimates on $G_{\tau p}(z,w)$, the fundamental solution of $\square_{\tau p}$.

Corollary 2. *Let $G_{\tau p}(z,w)$ be the integral kernel of the fundamental solution for $\Box_{\tau p}$. If X^α is a product of $|\alpha|$ operators of the form $X^j = \bar{Z}_{\tau p}, Z_{\tau p}$ if acting in z and $\overline{(\bar{Z}_{\tau p})}, \overline{(Z_{\tau p})}$ if acting in w, then there exists constants $C_{1,|\alpha|}, C_2 > 0$ so that if $\tau > 0$,*

$$\left|X^\alpha G_{\tau p}(z,w)\right| \leqslant C_{1,|\alpha|} \begin{cases} \log(\frac{2\mu(z,1/\tau)}{|z-w|}), & |z-w| \leqslant \mu(z,1/\tau),\ |\alpha| = 0, \\ |z-w|^{-|\alpha|}, & |z-w| \leqslant \mu(z,1/\tau),\ |\alpha| \geqslant 1, \\ \frac{e^{-C_2 \frac{|z-w|}{\mu(z,1/\tau)}} e^{-C_2 \frac{|z-w|}{\mu(w,1/\tau)}}}{\mu(z,1/\tau)^{|\alpha|}}, & |z-w| \geqslant \mu(z,1/\tau). \end{cases}$$

Also, C_2 does not depend on α.

Proof. We just need to integrate in s for the estimate. We first show the $|\alpha| = 0$ case. Let $\delta > 0$. Then if $c_2 = \frac{1}{32}$,

$$\int_0^\infty H_{\tau p}(s,x,y)\,ds \leqslant \int_0^\delta \frac{1}{s} e^{-c_2 \frac{|x-y|^2}{s}}\,ds + \int_\delta^\infty \frac{1}{s} e^{-c \frac{s}{\mu(x,1/\tau)^2}}\,ds = I + II.$$

To estimate I, we let $t = c_2 \frac{|x-y|^2}{s}$, so $-\frac{1}{t}\,dt = \frac{1}{s}\,ds$ and

$$I = \int_{\frac{|x-y|^2}{\delta}}^\infty \frac{1}{t} e^{-t}\,dt.$$

If $c_2 \frac{|x-y|^2}{\delta} \leqslant 1$, then

$$I = \int_{c_2 \frac{|x-y|^2}{\delta}}^1 \frac{1}{t} e^{-t}\,dt + \int_1^\infty \frac{1}{t} e^{-t}\,dt \leqslant C\left(\log\left(\frac{\delta}{|x-y|^2}\right) + 1\right).$$

Also, if $c_2 \frac{|x-y|^2}{\delta} \geqslant 1$,

$$I \leqslant \frac{1}{c_2 \frac{|x-y|^2}{\delta}} \int_{c_2 \frac{|x-y|^2}{\delta}}^\infty e^{-t}\,dt = C \frac{\delta}{|x-y|^2} e^{-c_2 \frac{|x-y|^2}{\delta}} \leqslant C e^{-c_2 \frac{|x-y|^2}{\delta}}.$$

To estimate II, set $t = c \frac{s}{\mu(x,1/\tau)^2}$, and we have

$$II = \int_{c \frac{\delta}{\mu(x,1/\tau)^2}}^\infty \frac{1}{t} e^{-t}\,dt.$$

If $c\frac{\delta}{\mu(x,1/\tau)^2} \leqslant 1$, we have

$$II = \int_{c\frac{\delta}{\mu(x,1/\tau)^2}}^{1} \frac{1}{t} e^{-t}\, dt + \int_1^\infty \frac{1}{t} e^{-t}\, dt \leqslant C\left(\log\left(\frac{\mu(x,1/\tau)^2}{\delta}\right) + 1\right).$$

Also, if $c\frac{\delta}{\mu(x,1/\tau)^2} \geqslant 1$,

$$II \leqslant \left(c\frac{\delta}{\mu(x,1/\tau)^2}\right)^{-1} \int_{c\frac{\delta}{\mu(x,1/\tau)^2}}^{\infty} e^{-t}\, dt = \frac{\mu(x,1/\tau)^2}{\delta} e^{-c\frac{\delta}{\mu(x,1/\tau)^2}} \leqslant Ce^{-c\frac{\delta}{\mu(x,1/\tau)^2}}.$$

Setting $\delta = \frac{|x-y|}{\mu(x,1/\tau)}$ yields the result. The $|\alpha| \geqslant 1$ case uses the same argument as the $|\alpha| = 0$ case, except that the on-diagonal estimate is simpler since

$$\int_\delta^\infty \frac{1}{s^{1+\frac{1}{2}|\alpha|}}\, ds \sim s^{-\frac{1}{2}|\alpha|}$$

converges. □

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Appendix A. A comparison of the estimates of $G_p(z, w)$

We will focus on homogeneous polynomials of the form $p_1(z) = |z|^{2m}$ and $p_2(x+iy) = x^{2m}$. We will write $z = x + iy$. Recall from above that if ρ_{p_j} is the metric in [4] associated to p_j, the $d\rho_{p_j}^2 \sim \mu_{p_j}(z,1)^{-2}\, ds^2$.

To show that the estimates in Corollary 2 and Christ's estimates in [4] of $\tilde{G}_p(z, w)$ agree, we must show that for $j = 1$ or 2,

$$\frac{|z-w|}{\mu_{p_j}(w,1)} + \frac{|z-w|}{\mu_{p_j}(z,1)} \sim \rho_{p_j}(z,w). \tag{A.1}$$

(A.1) will follow from Corollary A.2 and Proposition A.4 for both p_1 and p_2. In fact, the p_1 case shows that the estimates agree whenever $p(z)$ is a homogeneous, subharmonic polynomial of degree $2m$ whose Laplacian does not vanish on the unit circle. Also, in the case $m = 2$, an elementary computation shows that $p(z) = x^4$ is equivalent to the general case for subharmonic, nonharmonic homogeneous polynomials of degree 4.

Proposition A.1.

$$\mu_{p_1}(z,1) \sim \min\left\{1, \frac{1}{|z|^{m-1}}\right\}, \qquad \mu_{p_2}(z,1) \sim \min\left\{1, \frac{1}{|x|^{m-1}}\right\}.$$

This computation has the following immediate corollaries.

Corollary A.2.

$$\frac{|z-w|}{\mu_{p_1}(w,1)}+\frac{|z-w|}{\mu_{p_1}(z,1)}\sim |z-w|+|z-w|\big(|z|^{m-1}+|w|^{m-1}\big)$$

and

$$\frac{|z-w|}{\mu_{p_2}(w,1)}+\frac{|z-w|}{\mu_{p_2}(z,1)}\sim |z-w|+|z-w|\big(|\mathrm{Re}\,z|^{m-1}+|\mathrm{Re}\,w|^{m-1}\big).$$

Corollary A.3.

$$\frac{1}{\mu_{p_1}(z,1)^2}\sim 1+|z|^{2m-2},\qquad \frac{1}{\mu_{p_2}(z,1)^2}\sim 1+|x|^{2m-2}.$$

At this point, we will concentrate on sketching a computation of $\rho_{p_1}(z,w)$. The computation for $\rho_{p2(z,w)}$ is analogous and shows

$$\rho_{p_2}(z,\zeta)\sim |z-\zeta|+|z-\zeta|\big(|\mathrm{Re}\,z|^{m-1}+|\mathrm{Re}\,\zeta|^{m-1}\big).$$

Proposition A.4.

$$\rho_{p_1}(z,\zeta)\sim |z-\zeta|+|z-\zeta|\big(|z|^{m-1}+|\zeta|^{m-1}\big),$$

where the constant depends only on m.

Proof. As a consequence of Corollary A.3,

$$\rho_{p_1}(z,\zeta)\sim \inf_{\alpha}\left\{\int_0^1\big(1+\left|\alpha(t)\right|^{m-1}\big)\left|\alpha'(t)\right|dt\right\}. \tag{A.2}$$

If $\alpha(t)=z(1-t)+\zeta t$, then

$$\int_0^1\big(1+\left|\alpha(t)\right|^{m-1}\big)\left|\alpha'(t)\right|dt\lesssim |z-\zeta|+|z-\zeta|\big(|z|^{m-1}+|\zeta|^{m-1}\big).$$

For the other direction, we give a more complete argument.

$$\rho(z,\zeta)\sim \inf_{\alpha}\left\{\int_0^1\big(1+\left|\alpha(t)\right|^{m-1}\big)\left|\alpha'(t)\right|dt\right\}\gtrsim |z-\zeta|+\inf_{\alpha}\left\{\int_0^1\left|\alpha(t)\right|^{m-1}\left|\alpha'(t)\right|dt\right\}.$$

Now, set $\gamma(z,\zeta)=\inf_\alpha\{\int_0^1 |\alpha(t)|^{m-1}|\alpha'(t)|\,dt\}$. Then

$$\gamma(z,\zeta)=\inf_\alpha\left\{\int_0^1\left|\frac{d}{dt}\alpha(t)^m\right|dt\right\}\geqslant \inf_{\substack{\beta\\ \beta(0)=z^m\\ \beta(1)=\zeta^m}}\left\{\int_0^1\left|\beta'(t)\right|dt\right\}=\left|z^m-w^m\right|.$$

Thus, if $|z|\geqslant 2|\zeta|$, then $|z^m-\zeta^m|\sim|z^m|\sim|z-\zeta|(|z|^{m-1}+|\zeta|^{m-1})$. Also, note that there is a one-to-one correspondence of paths between z and ζ and paths between rz and $r\zeta$ by sending $\alpha(t)$ to $\alpha_r(t)=r\alpha(t)$. Hence, it follows immediately that $\gamma(rz,r\zeta)=r^m\gamma(z,\zeta)$, so without loss of generality, we can assume that $|z|=1$ and $\frac{1}{2}<|\zeta|\leqslant 1$. Let $\zeta/z=re^{it}$. Note that $t=\arg\zeta-\arg z$. Also, we can write

$$\left(z^m-\zeta^m\right)=z^m\left(1-r^me^{imt}\right)=z^m\left(1-re^{it}\right)\prod_{k=1}^{m-1}\left(e^{2\pi i\frac{k}{m}}-re^{it}\right).$$

Note that if $|t|<\pi/m$, then $|e^{2\pi i\frac{k}{m}}-re^{it}|>c>0$ for some constant c and for all $k=1,2,\dots,m-1$, and in this case

$$\left|z^m\left(1-re^{it}\right)\prod_{k=1}^{m-1}\left(e^{2\pi i\frac{k}{m}}-re^{it}\right)\right|\geqslant c|z-\zeta||z|^{m-1}\geqslant\frac{c}{2}|z-\zeta|\left(|z|^{m-1}+|\zeta|^{m-1}\right).$$

Finally, if $|t|\geqslant\pi/m$, then $\rho(z,\zeta)\geqslant c$ for some constant $c>0$, and this is the desired result as $|z-\zeta|\geqslant c_1$ and $|z|,|\zeta|\in[\frac{1}{2},1]$. □

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